The Computer Museum
Museum Wharf
300 Congress Street
Boston, MA 02210

Opening Reception

"Colors of Chaos"

THE COMPUTER MUSEUM

presents

"Colors of Chaos"

William C. Norris Gallery, 5th floor

An exhibit of spectacular computer graphics based on research in complex dynamics, including Julia Sets and the Mandelbrot Set.

Join us on Thursday, April 10, 1986

6:00-10:00 p.m. Wine & Cheese Reception
7:00-8:00 p.m. Lecture: Professor Robert Devaney

Professor Devaney, chairman of Boston University's Department of Mathematics, whose graphics are featured in the exhibit, will describe how these images are generated and what they mean.

Members admitted free; non-members $4.00

R.S.V.P. by Monday, April 7th
Kathy Keough, The Computer Museum (617) 426-2800, ext. 340

Dear Mr. Strimpel:

Here is the little demo disk that I promised you. Start it up by entering DISPLAY, when the disk is in the default drive

Stop it by entering "X" from the keyboard.

The program runs best when run from a RAM or hard disk. First transfer all files to the disk, make it the default, then enter DISPLAY.

Please call or write with your comments!

Mark

Mark Bolme (206-455-4130)
Sintar Software
P.O. Box 3746
Bellevue, WA 98009


DEMO - 3/12/86
"x" to exit

Welcome to the opening of our new exhibit Colors of Chaos.

The exhibit is about the field of complex dynamics, a subject of interest both to mathematicians and physicists. The advent of high-speed computers and color graphics has resulted in the ability to perform experiments and discover new patterns and structure in this field. This has injected a great burst of enthusiasm into the subject.

Another major stimulus was the discovery a few years ago of the Mandelbrot set. I won't say more about this now as it is featured in the exhibit and I'm sure Professor Devaney will be talking about it. We first saw glimpses of the astonishing structure of this set when Benoit Mandelbrot gave a lecture here just over a year ago. He showed highly detailed spectacular maps of the Mandelbrot set generated by a group of physicists at the University of Bremen in West Germany. As luck would have it, one of the leaders of this group, Peter Richter came as a visitng professor to Boston University last year. We contacted him and he was willing to help us put together an exhibition with his group's images. We also found out about Robert Devaney's related work. The Bremen group had concentrated on Mandelbrot sets and Julia sets of the squaring process and for models of magnetization. Robert had worked on the corresponding pictures for the trigonometrical functions sine, cosine and exponential. The two were married, and you see the result today.

We are extremely pleased to have Robert Devaney with us here tonight. Peter Richter's visit to the US came to an end last month and he is now back in Bremen, West Germany, so he was not able to be here.

Robert graduated from Holy Cross College in 1969 and received a PhD from the University of California Berkeley in 1973. He has been on the mathematical faculty of Northwestern University, Tufts, and the University of Maryland. In 1980 he joined Boston University and is now Chairman of the Mathematics Department there. He has been interested in dynamical systems for many years, since well before the subject became fashionable. He has published over 20 papers on the subject, and has written two books. The more recent one, entitled Chaotic Dynamical Systems is on sale at our store downstairs.

So let me now hand over to Professor Robert Devaney to tell us about the Colors of Chaos!

COLORS OF CHAOS

A Special Exhibit   April 11 - September 8 1986

by Oliver Strimpel

The pictures of Colors of Chaos are the results of using computer graphics as a tool for research in complex dynamics, a branch of mathematics and physics. The goal is to understand what happens to simple mathematical formulae when they are iterated.

The pictures are built up in the following way: each point is iterated using the mathematical formula under investigation. For example, if the formula is the trigonometrical function cosine, this corresponds to entering the number corresponding to a point in the picture into a calculator, and pressing the cosine button again and again. The point in the picture is then colored depending on what happens. In some of the pictures, the color shows how quickly the point "escapes" to infinity, leaving black those points that never escape. In others, the colors show where points end up under iteration, with the shades indicating how quickly they get there. Thus the colors represent the dynamics of the iteration.

Julia Sets and Mandelbrot Sets

Two types of picture can represent the iterative process. In the Julia set, the initial value of the complex number at the start of the iteration is varied over the plane. The parameters of the iteration are fixed. A point is in the set if it lies on the boundary of the points that become larger and larger as the iteration proceeds. Each set of parameters creates a whole different Julia set. One of the remarkable discoveries revealed in Robert Devaney's images is that the Julia set can change dramatically, even evaporate completely, for very small changes in the parameters of the iteration. The picture shown on the front cover is a still image from a film showing the dramatic change in the structure of the Julia set for the sine function as the parameter is varied. The black region shows points that have not escaped to infinity after 35 iterations, while the colored regions show escaped points. Red points tend to infinity the fastest, followed by points colored in orange, yellow, green, blue and violet.

The Mandelbrot set is an example of the second type of picture. Here, it is the value of the parameter that is varied over the plane and the initial value of the complex number is set to zero everywhere. Each formula being iterated has only one of these pictures. A point is a member of the set if it never escapes to infinity under iteration.

The first formula investigated by Benoit Mandelbrot in 1975 was simply the squaring of the complex number in which one iteration step consists of squaring the number and adding a constant. By varying this constant over the plane as the parameter, Mandelbrot discovered ~~the~~ a ~~characteristic~~ cardioid shaped set ~~that is~~ with a hairy boundary ~~normally what is meant by~~ the ~~term~~ Mandelbrot set. To the mathematicians' surprise, this shape appears to be universal in that it crops up, albeit somewhat modified in detail, when many other formulae are iterated. When the boundary of the Mandelbrot set is examined in fine detail, baroque swirls, spirals and tendrils appear, including some that lead to offshoots containing smaller replicas of the Mandelbrot set itself. It is this fascinating structure at the boundary of the Mandelbrot set that is vividly represented in the Colors of Chaos images that came from the Bremen group.

Julia sets and Mandelbrot sets can take a lot of computing. Firstly, each point of the picture has to be iterated separately (unless one uses a parallel machine), so the time taken to create an image is proportional to the total number of pixels computed. Secondly, the number of iteration steps required per point can be as high as several thousand. The closer to the boundary of the Mandelbrot or Julia set you go, the longer it takes a point to 'make up its mind' as to where it is really attracted. Each iteration step takes several floating point multiplies or the evaluation of a tigonometrical function. Robert Devaney has just used 72 hours of the Cray supercomputer at Digital Productions to make a new spectacular film showing Julia sets of cosine. It will be added to the video showing in the exhibit.

Images in the Colors of Chaos Exhibit

Can be moved

A series of twelve pictures shows Julia Sets and Mandelbrot Sets generated by the iteration of polynomial functions and ratios therof by a team from the University of Bremen led by Heinz-Otto Peitgen and Peter Richter.

A second series shows Julia sets of sine, cosine and the exponential function by Robert L Devaney from the Department of Mathematics at Boston University.

Why do this?

Because it's there! The beauty of the images continues to spur along ever more detailed explorations of these newly discovered objects. But the computation of Julia sets and Mandelbrot sets can also be viewed as numerical experiments in complex dynamics. When combined with mathematical intuition, they uncover universal patterns and stimulate the progress of mathematics. They are also important in the new field of fractal geometry. Indeed the boundary of the Mandelbrot set is a fractal. Mandelbrot conjectures that it may have a fractal dimension of 2, which would mean that all offshoots would have to be connected to the main set and that we have barely scratched the set's surface! According to John H. Hubbard, mathematician at Cornell University and the first to make detailed computer generated images of it, the Mandelbrot set is "the most complicated object in mathematics".

The iteration of complex functions is also of interest because it models the way many non-linear natural systems evolve. Simple iterative laws can predict very complex, chaotic behaviour. Examples include the growth and decline of the population of a biological species, the motions of the planets, the changes in the weather and even the daily fluctuations of the Stock Market.

Further Reading:

The Beauty of Fractals by H-O Peitgen and P H Richter, Springer-Verlag 1986
This new release contains approximately 75 color and 65 black and white illustrations, including many of the images on display in the exhibit. The text appeals to both layman and expert, and ranges

from philosophical background to suggestions on how to generate your own fractal images.
($35 postpaid, $31.50 members)

The Fractal Geometry of Nature by Benoit B Mandelbrot, W H Freeman 1983
($39.95 postpaid, $36.45 members)

Introduction to Chaotic Dynamical Systems by Robert L Devaney, Benjamin Cummings 1985
($34.95 postpaid, $31,95 members)

The above books are available from The Computer Museum Store. Also available are a set of 8 color postcards of the Bremen images, including several on display in the exhibit ($4.00 + 1.00 postage).

Scientific American Computer Recreations column by A K Dewdney August 1985 issue

(To go inside front cover)

Front Cover

Colors of Chaos

Julia set of (.1+.17i)sin(z) after 35 iterations by Robert L Devaney, Boston University. (see

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article on page 16)

Caption

The Mandelbrot set, courtesy of ~~John Milnor~~ Benoit Mandelbrot / IBM.

The Colors of Chaos

Robert L. Devaney
Department of Mathematics
Boston University

Peter Richter
Department of Physics
Boston University, and
University of Bremen

Heinz-Otto Peitgen
Department of Mathematics
University of Bremen, and
University of California, Santa Cruz

This exhibit features a variety of computer graphics images which arise in the field of mathematics known as complex dynamics. The goal of researchers in this field is to understand the behavior of simple mathematical expressions when they are iterated. Typically, all points in a grid or lattice in the plane will be iterated; the points are then color-coded depending upon their fate under iteration. The resulting patterns on the graphics screen allow the researcher to analyze the dynamics of the mathematical expression. For example, one might assign a color to each point depending upon how quickly its iterates "escape" to infinity, leaving black the points which never escape. Alternatively, one might use colors to differentiate between points which have different fates under iteration, using shades to indicate how quickly points tend to their ultimate fate. We have used both of these "algorithms" in this exhibit.

Besides their importance to researchers in dynamics, these pictures also play a role in the emerging field known as fractal geometry. The complex patterns generated by the boundaries of regions with different colors are fractals. Their geometric structure is also an important subject of contemporary research. Here we see that even the simplest mathematical expressions generate fractals of amazing complexity and beauty.

FRACTALS

Many of the Julia sets in these pictures are "fractals". Without resorting to a precise mathematical definition of this term, suffice it to say that a fractal is an object which tends to look the same under successive magnifications. Note the picture of the Julia set of $\sin z$ and its magnification. Basically, the black regions resemble "infinite snowmen." In the enlarged picture, we see that these snowmen occur in miniature in every direction in the picture.

ITERATION OF SIN(Z)

The complex function $\sin z$ has a rather interesting structure when thought of as a dynamical system, i.e., when iterated. These pictures detail the dynamical behavior of $\sin z$ as well as $(1 + \lambda i) \sin z$ for various values of the parameter λ. The colored regions in these pictures indicate points which "tend" to ∞ under iteration of these maps. The black regions contain those points which remain bounded under iteration. One may show that the colored regions are actually the Julia sets of the maps and so contain all of the chaotic dynamics, while the black regions are the stable sets. Note how the black region for $\sin z$ (the first picture) explodes into color as the parameter is varied.

The colors in these pictures simply tell how quickly a point tend to ∞ under iteration. Red points tend to ∞ most quickly, followed by points colored in shades of orange, and then followed by yellow, green, blue, and violet.

Robert L. Devaney

DYNAMICAL SYSTEMS

Dynamical Systems is a branch of Mathematics which studies the evolution in time of discrete or continuous processes. Examples of discrete processes include the growth and decline of the population of a biological species or the daily fluctuations of the Stock Market. Examples of continuous processes include the motion of the planets or the changes in the weather. The Mathematician is interested in understanding the long term behavior of these processes, i.e., will the population of the species tend to die out or will it explode in the far distant future?

To study such processes, the Mathematician often constructs simplified mathematical models of the physical systems, and then studies these models instead. In these pictures, we have tried to detail the dynamics of several extremely simple dynamical systems: iteration of the complex analytic functions $\sin z$, $\cos z$, and $\exp z$.

THE JULIA SET

These pictures describe the barest outline of a mathematical object called a Julia set, after the French mathematician, Gaston Julia, who first studied these sets in the early twentieth Century. For functions like $\sin z$ (as depicted here), or $\cos z$, or $\exp z$, the Julia set is the closure of the set of points which tend to ∞ under iteration of the function. Equivalently, the Julia set may also be characterized as the set of expanding "periodic points." As you can see, the topological structure of these sets is quite intricate. For the mathematician, understanding the structure of these sets is a basic goal: since these functions are among the most elementary transcendental functions, a detailed knowledge of their Julia sets is a necessary first step in the understanding of the more complicated equations that occur in nature.

COLORS OF CHAOS

The pictures shown here are the results of using computer graphics as a tool for research in complex dynamics, a branch of mathematics and physics. The goal is to understand what happens to simple mathematical formulae when they are iterated. Iteration means evaluating a formula again and again, each time using the previous value of the formula as the new value of the variable. In physics, this process describes the way natural systems evolve.

These computer experiments, combined with mathematical intuition, uncover universal patterns and stimulate the progress of mathematics. They are also important in the new field of fractal geometry (described in The Computer and the Image gallery) and in the understanding of nonlinear processes in nature.

The pictures are built up in the following way: each point in the picture is iterated using the mathematical formula under investigation. For example, if the formula is the trigonometrical function cosine, this corresponds to entering the number corresponding to a point in the picture into a calculator, and pressing the cosine button again and again. The point in the picture is then colored depending on what happens. In some of the pictures, the color shows how quickly the point "escapes" to infinity, leaving black those points that never escape. In others, the colors show where points end up under iteration, with the shades indicating how quickly they get there.

It takes a lot of computing to produce a high resolution image because each point is iterated up to several thousand times. Indeed, Robert Devaney is now using a Cray supercomputer. A low resolution image can be made in a few hours on a personal computer.

The images form two series:

Julia Sets and Mandelbrot Sets generated by the iteration of polynomial functions and ratios therof by a team from the University of Bremen led by Heinz-Otto Peitgen and Peter Richter.

Iterations of sine, cosine and the exponential function by Robert L Devaney from the Department of Mathematics at Boston University.

Further Reading

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COLORS OF CHAOS

relativity

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physics.

This process corresponds to evolution in time of physical systems in nature

(Ordinary time development of a system is physics.)

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thousand

The images form two series:

generated by the iteration of polynomial functions and ratios ~~thereof~~ of polynomial functions (rational functions).

and

Mandelbrot Sets, Julia Sets, and iterations of polynomial functions by a team from the University of Bremen led by Peter Richter and Heinz-Otto Peitgen.

2 sentences

Iterations of sine, cosine and the exponential function by Robert L Devaney from the Department of Mathematics at Boston University.

Further Reading

The Beauty of Fractals by H-O Peitgen and P H Richter, Springer-Verlag 198~~5~~6

The Fractal Geometry of Nature by Benoit B Mandelbrot, W H Freeman 1983

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Scientific American by A K Dewdney August 1985 issue

27 February 1986

Jeff Michell
Graphic Art Framing
300 Harvard St
Brookline
MA 02146

Dear Jeff

Please find enclosed cheque for $322.51 to cover framing of the pieces we brought in about 3 weeks ago. The remaining pictures will be picked up within the next few days either by Peter Richter or Mary Olmstead of Boston University.

Please let me know if there is any problem.

Thank you.

Yours sincerely

Oliver Strimpel
Curator

→ Olivia

Frontiers of Chaos (to be renamed)

Proposal for a Temporary Exhibit

Introduction

Frontiers of Chaos is a graphic exhibit of some 50 images that has been replicated by the Goethe Institute and is touring in Europe and America, booked till well into 1987. I saw it first in the Exploratorium in July during SIGGRAPH. The Boston Science Museum has booked it for January 1987.

The images were computed by a group at the University of Bremen, one of whom (Peter Richter) is visiting BU this year. They show computer graphic renderings of Mandelbrot Sets, Julia Sets and some other mathematical functions. They are all based on a similar type of idea, that of iteration of a complex function. The images can appeal on several levels:

1. Pretty pics
2. How can I do these on my pc?
3. What's the meaning/maths behind this?

Content of CM Exhibit

Peter Richter happens to have 14 prints (unframed), many of which are in the FOC exhibit, all of which are interesting.

The Chairman of the maths dept at BU, Robert Devaney has also produced similar images (iterating exp, cos, sin as opposed to the squaring function) which, though not as well colored or as sharp as the Bremen images, are still good enough to show. He also has a 2 minute 16mm film clip showing what happens as you change parameters - beautiful patterns that mimic the onset of turbulence.

I propose we put together a joint exhibition consisting of

1. 14 Bremen images from Peter Richter
2. approx 14 of Robert Devaney's pictures
3. Video tape of Robert Devaney's film
4. Something interactive, possibly on the Amiga showing generation of Julia sets using graphics and sound. This could then replace/supplement the IBM pc fractal demo in the image gallery.

The exhibit would be introduced with several text panels. Each image would need a small label, along the lines of the Byte covers.

Timescale and Budget

Open April 15 approx and close if/when SIGGRAPH '85 art show comes, but not sooner than July.

Costs:

Framing	500	
Text panels	600	
Video tape	200	(use existing player, control box and monitor)
Interactive	500	
Miscellaneous	200	
Total	2000	fr 120 budget

OS
11/21/85

Oliver

looks good

goals:

1) Fixed opening date of the show by Jan 15, & info to Locize to allow advanced promo work — possible color cover for some publication.

2) museum to retain prints (if possible) after framing, or

3) produce new prints for framing then donate prints — these could be good for a color lab.

go ahead

FOR IMMEDIATE RELEASE
MARCH 17, 1986

PRESS CONTACT: LOUISE DOMENITZ
(617) 426-2800

EXHIBIT OPENING AT THE COMPUTER MUSEUM

"COLORS OF CHAOS"
The Art of Dynamical Systems
William C. Norris Gallery, 5th Floor

"COLORS OF CHAOS" ON DISPLAY APRIL 10 - SEPTEMBER 9, 1986

An exhibit of spectacular computer graphics based on research in complex dynamics, including Julia Sets and the Mandelbrot Set. These vividly colored images are displayed as enlarged photographs, on video tape, and running on a computer.

Special Opening Lecture
Thursday, April 10th at 7:00 p.m. in the 5th floor auditorium: Professor Robert Devaney, Chairman of Boston University's Department of Mathematics, whose graphics are featured in the exhibit, will describe how these images are generated and what they mean.

About the exhibit:
The **"Colors of Chaos"** exhibit is special because despite the simple nature of the rules used to build the pictures, exceedingly rich and varied shapes result. While the theories touch on some deep areas of mathematics, these beautiful images can be appreciated on an aesthetic level. In fact, their beauty spurred mathematicians to explore them further as a purely artistic endeavor. Scaled down versions can be generated by anyone equipped with a personal computer with some graphics capability.

"Colors of Chaos" may offer school children a new perspective on mathematics, by showing its subtlety and beauty. These computer graphics are equally fascinating as tools for science and as products of art.

These graphics stem from the research of mathemetician Benoit B. Mandelbrot, who recently found a set of points in the complex plane that has phenomenal and totally unexpected richness of structure. The boundary of this fascinating entity, named the Mandelbrot Set, is a fractal.

The formula is so simple, yet the myriad shapes that the boundary of the set takes are just being discovered. The computer can be used as a microscope on this world, zooming in to reveal newer structures at every level as finer detail is examined.

Since fractals were "rediscovered" in the mid-1970's, there has been a great interest in this new branch of mathematics and in the use of fractals in creating synthetic environments for film-makers and advertising video artists.

The Mandelbrot Sets were investigated by a group from the University of Bremen. The trigonometrical functions are by Robert Devaney of Boston University.

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THE COMPUTER MUSEUM is open Tuesday, Wednesday, Saturday and Sunday from 10 - 6; Thursday and Friday from 10 - 9. Admission is $4.00 for adults, $3.00 for students and senior citizens. The Museum is located at Museum Wharf, 300 Congress Street, Boston. Call the talking computer at (617) 423-6758 for more information on programs and membership.

The Computer Museum is a non-profit organization under IRC 501 (c) 3

The Mandelbrot Set

Benoit B. Mandelbrot discovered the set that carries his name in March 1980. He was studying the nature of boundaries that arise between territories when a number of centers compete for dominance. In mathematics, such competition can be modeled in the following way. Let every point of a plane - i.e., every pixel on our graphics terminal - find out where it belongs by entering a repetitive process in which the centers exert their attractive influence. After a number of steps, most points will be driven towards one of the competing centers. The exception are those which lie on the boundary between the various domains of attraction. One might guess that such boundaries are simple when the generating process is, but surprisingly this is not at all the case.

In Mandelbrot's case the process is as simple as can be. You choose a number c. Then to find the fate of a point x on the plane, take the square of x and add c - that's all for one step. Take the new point and repeat the process over and over again until it becomes apparent where it eventually leads to. Of course, only a computer is patient enough to do such stupid things for a grid of, say, a million different points. But then the result is quite amazing.

The twelve pictures surrounding the central figure - called Mandelbrot's Set - are examples of boundaries generated in this way. There are in fact no more than two competing centers in Mandelbrot's process: one at an infinite distance, the other somewhere near the center of the pictures (see the black dots in some of them). The second center need not exist, however, and that's exactly what Mandelbrot's Set is all about. It surveys the possible values of the numbers c for their impact on the nature of the corresponding boundaries.

The basic distinction is between connected and disconnected boundaries. When c is inside the Mandelbrot figure, then the corresponding boundary is connected, and it divides two territories from each other. When c is outside the Mandelbrot Set, then the boundary is merely a "dust" of isolated points which obviously cannot surround a territory; so these are cases where only one center exists. It dominates the whole plane except for the boundary points which may be viewed as the dissidents that do not like to be drawn to the center.

The boundary of Mandelbrot's Set is therefore the locus where those other boundaries change their nature from being connected to being disperse. Its own structure is almost unbelievably complex. Some of the best mathematicians in the world have just begun to unravel its secrets (among them John Milnor of Princeton who kindly provided the Mandelbrot Set shown here). They all agree that this bizarre object has a deep significance for our understanding of nonlinear processes.

In poster

The Mandelbrot Set

Benoit B. Mandelbrot discovered the set that carries his name in March 1980. He was studying nonlinear feedback processes in the complex plane. The simplest such process is the following: take a fixed number c; then choose a complex number z as input; compute the output z*z+c and use this as the next input. The question is then: where does this process lead to after many iterations?

The answer is simple if c=0. Then all numbers z inside the unit circle will eventually be driven towards zero, while numbers z outside the unit circle will grow indefinitely in magnitude. We say: the process has the two "attractors" zero and infinity; these attractors divide the plane among themselves, the unit circle being the boundary between their respective domains of attraction.

But what if c is not zero? The plate entitled "Julia Sets of the Mandelbrot Family" shows twelve examples of what may happen to the boundary. Obviously, the circle undergoes severe transformations. In the simplest possible case (example 1), it only gets deformed but still divides just two connected territories, the domains of attraction of infinity and of the black dot in the interior. Note, however, that it has lost its smoothness. When it is put under a microscope it looks much the same and does not lose its ruggedness. This property of "self similarity with structure on all scales" is typical for "fractals". It was established as a mathematically well defined feature of boundaries in a broad class of iteration processes by Gaston Julia (1920). Such boundaries are therefore known as Julia sets.

For other choices of c the process generates more complex boundaries. In the example 3 the inner attractor is no single point any more. For initial points inside the figure, the process never comes to rest. It converges towards the three black points and visits them in a cyclical fashion. The boundary in such a case is made up of infinitely many deformed circles.

Example 8 shows a case where a single point attractor is about to split up into a cycle of period 5. Such "parabolic" choices of c tend to produce particularly nice and complex boundaries. The pictures 7 and 12 are other examples.

Example 9 shows a "Siegel disc", named after the German mathematician Carl Ludwig Siegel who proved their existence in 1942. The same case is displayed in colors as picture BC1 of the exhibit. The disc containing the black dot absorbs all points from the other leaves, but when the process has arrived in this disc, it does not converge towards the center. Rather the points of the disc rotate around the center along the circles drawn. They proceed by an angle of 137.5 degrees in each step of the process. This is the Golden Angle which also characterizes the arrangement of successive leaves on a pine cone or an artichoke.

Another possibility is shown in the example 4. Here the inner attractor has disappeared. The "dendrite" has no interior. However, it still is a boundary of the domain of attraction of infinity. Points lying on this boundary will never leave it, although they tend to hop around on the dendrite in an erratic fashion. Example 10 is a combination of a dendritic structure with a period 4 attractor (see the little blobs sitting on the dendrite).

Finally, examples 5, 6, and 11 show "Fatou dust". Again there is no attractor besides infinity; the boundary has now disintegrated into a set of disconnected points.

Given that much variety in the boundary structure of the Mandelbrot process, the question arises how this behavior might be ordered. An old mathematical theorem by Pierre Fatou (1919) helped finding the answer. It was put to use by Benoit Mandelbrot in the following famous experiment: Choose a number c in the complex c-plane. Take z=0 as input to the Mandelbrot process with that chosen c. Let the process run until it becomes clear whether z escapes to infinity or not. If it does then assign the color white to the chosen number c; if it does not then let that c-value be black. Test all values c on a sufficiently dense grid of points, and the result will be the Mandelbrot set in black, see above.

That the fate of the point z=0 contains all the relevant information on the nature of the Julia sets is the truly remarkable essence of Fatou's theorem. But Mandelbrot's discovery brought to light a fine structure that Fatou could never have guessed. The Mandelbrot set has a main body with a system of buds, antennae, and satellite Mandelbrot sets that have a precise meaning in relation to the structure of the z-plane boundaries. The arrangement of the Julia sets discussed above gives an idea of this relationship.

Another remarkable feature is expressed in terms of the array of lines in the picture below. The closed lines surrounding the Mandelbrot set are lines of equal escape time for the point z=0, in the experiment based on Fatou's theorem. But John Hubbard (Cornell) and Adrien Douady (Paris) have shown that these same lines may be interpreted as equipotential lines of an electrically charged metallic Mandelbrot set! The lines perpendicular to these would then be the corresponding electric field lines. Physicists can only be amazed at the fact that the electrostatic problem of Mandelbrot's set is easier to compute than that of an ordinary polygon.

There is a fascinating unsolved problem associated with the electric field lines. Consider the lines far away from the center where they radiate out as straight lines, at an angle which is a certain multiple of 360 degrees. If that multiple is a rational number, then mathematicians can tell from which point on the Mandelbrot set the corresponding line originates. But if it is irrational, then it is not even known whether there is a definite point of origin on the boundary of the Mandelbrot set.

Pictures from the Mandelbrot family

The pictures BC1 - BC6 illustrate the rich complexity of the Mandelbrot process.

BC1: Siegel disc. For c = -.39054 -.58679 i the domains of attraction of the process are shown in the z-plane. The greenish outside is attracted towards infinity, and the color grading indicates how often a point must be iterated to grow beyond 100 in magnitude. All points inside the red figure are first drawn into the big disc at the lower left center, and then they keep rotating around that center along the circles drawn.

BC2 - BC6: Details from the boundary of the Mandelbrot set. The introductory text on the Mandelbrot set explains that the boundary of the Mandelbrot set is the locus in the c-plane where the corresponding z-plane boundaries change from being connected to being disconnected. This boundary is therefore of particular interest for an understanding of the process. It also displays the most intricate forms, as these pictures demonstrate. The following guiding picture identifies the locations on the Mandelbrot set where the five blow-ups have been taken. Some go into tenthousandfold magnification.

Julia and Mandelbrot sets in the theory of magnetism

In 1983, the French physicists B. Derrida, L. DeSeze, and C. Itzykson took up an old suggestion of C.N. Yang and T.D.Lee who in 1952 proposed to describe the magnetic phase transition in terms of certain singularities in a complex temperature plane. The idea did not lead very far at the time as it turned out that computation of those singularities was exceedingly difficult; it was almost forgotten when L.P. Kadanoff and K.G. Wilson solved the problems of the theory of phase transitions by developing the renormalization theory.

The beautiful twist in the work of Derrida et al. was to show that in certain models of magnetism, the concept of Julia sets, when applied to the renormalization process, leads directly to the Yang-Lee singularities. The picture BC7 is an example of a complex phase diagram in the spirit of Yang and Lee. The white spots identify "temperatures" where the given material would be magnetic. They sprinkle the nonmagnetic temperature range, the boundary being the Julia set of the temperature renormalization.

Picture BC8 is a detail from a parameter study of the model of magnetism that produces Julia sets as shown in BC7. The occurance of the Mandelbrot set in a context quite different from the original one suggests a universality that is only partially understood at this time.

Newton's root finding algorithm

The classical examples of Julia sets are taken from Newton's root finding algorithm. Given a polynomial equation f(x,y)=0 in an x,y-plane, the algorithm allows one to find a root from any odd guess, provided the guess does not happen to fall on the boundary between the "domains of attraction" of different roots. It is of great practical interest in numerical mathematics to identify the structure of these "basins" of the roots and of their boundaries. (An issue that incidentally also leads to Mandelbrot's set is the question under what condition the competition of the roots may leave room for additional attractors unrelated to the roots of the given equation.)

The individual balls of picture BC9 give different views on the complex plane (represented as a Riemann sphere with infinity at the north pole) on which the basins of the roots of a third order polynomial are indicated by three different colors. The boundary structure is much more complicated than might have been expected for a simple competition process. In fact, Julia established the following odd property: each boundary point must at the same time be bordering on all three territories! We are used to such points as exceptional cases only, like Basel bordering on Switzerland, France, and Germany. The picture demonstrates how boundaries may be constructed exclusively from such points. Of course, such boundaries have to be fractals.

In picture BC10 the Newton algorithm is applied to root finding in a real x,y-plane. Again the colors identify the basins of different roots. They reveal another kind of entanglement than in the complex case. There is no classical theory for these processes, but guided by computer experimentation, H.-O. Peitgen and M. Pruefer at Bremen and K. Schmitt at Salt Lake City have collaborated to develop a theoretical understanding. Picture BC11 is a corresponding parameter study most features of which are as yet unexplained.

Discretized Lotka-Volterra competition

Whenever differential equations are solved on a digital computer they need to be discretized and integrated step by step. It is well known that the nature of their solutions may be considerably altered by this requirement. An important chapter of numerical mathematics is therefore to study these modifications. Picture BC12 is the result of such a study, performed on the classical Lotka-Volterra system that describes the oscillations in populations of predators and their prey.

The population size of the predators is plotted in the vertical, that of the prey in the horizontal direction. The differential equations are known to yield closed lines that are concentrically arranged about a marginally stable steady state solution. The effect of the discretization shown here is quite dramatic. For initial points in the grey area, the process is unstable and blows up to infinity. Points inside the reddish region are sucked up by an attracting limit cycle. The color grading is used to visualize the internal structure of its basin.

The Bremen Pictures

Twelve pictures of this exhibition were produced in the computer graphical lab of the Center for Complex Dynamics at the University of Bremen, Federal Republic of Germany. The Center is headed by professors Heinz-Otto Peitgen (currently at the Department of Mathematics, University of California at Santa Cruz) and Peter H. Richter (visiting professor at the Department of Physics, Boston University).

The lab is being managed by Dr. Hartmut Juergens and Dr. Michael Pruefer. The software for the picture production was developed by Dr. Dietmar Saupe and Dr. Michael Pruefer. It will be described in the forthcoming book "The Beauty of Fractals", Springer-Verlag, Heidelberg 1986.

The pictures were produced in the spring of 1985 as part of the exhibit "Frontiers of Chaos" which is currently touring the United States under the auspices of the Goethe Institute. This exhibit of some 90 objects will be displayed at the Boston Museum of Science from January through May 1987.

As of early 1985, the lab was equipped with

- a 16 bit NORD 100 computer of NORSK DATA
- a Tektronix 4014 black and white terminal with hardcopy unit
- a PS-300 vector-refresh system for 3-D graphics (Evans and Sutherland)
- an AED 767 color graphics system with Mitsubishi monitor (575x768 8 bit pixels)
- a matrix camera MATRIX 3000 (Honeywell)
- a U-Matic video recorder

Sept 26 1986 Phone memo

Spoke to Nancy Hollomon.
Fine by her to extend the exhibit
till end of ~~Q~~ September.
OS to resolve purchase by CM of
pictures by ~~and~~ directly with
Peter Richter.
Olivi 8/26/86

Spoke to 'Sally', at Phys sci
will call me back when Larry Sulak
gets back

Spoke to Rob Devaney 8/28
Said OK till Oct 1.
OS said he'll remove them by
Oct 1.

The
Computer
Museum

300 Congress Street
Boston, MA 02210

(617) 426-2800

April 25 1986

Peter Richter
Fachbereich Physik
Universitat Bremen
D-2800 Bremen 33

Dear Peter

We had an excellent opening for Colors of Chaos. The pictures look really excellent here. We put the Mandelbrot set series on the green wall and this brings out the reddish colors very well indeed. Over a hundred people attended and seemed impressed and enthusiastic about the pictures. Robert's lecture went very well too - no doubt you already know what a good lecturer he is.

It as a pity you couldn't be there, but I think you would have been pleased with the way it went, and will like the exhibit.

Incidentally, the post cards are selling very well indeed.

The physics department has agreed to waive the charge for the Mandelbrot set posters in exchange for us waiving the framing costs. We have a number of people interested in buying images and will have no difficulty selling the last two pictures. Indeed some have suggested that we could do a good business reproducing and selling them in the store. Is anyone already doing this, or is this being discouraged so as not to undercut the Goethe Institute?

Thank you again for everything. I hope you are settled in again by now. I look forward to seeing you during your next visit.

Yours sincerely

Oliver Strimpel

The
Computer
Museum

300 Congress Street
Boston, MA 02210

(617) 426-2800

March 28 1986

Peter Richter
Fachbereich Physik
Universitat Bremen 33
W Germany

Dear Peter

Thank you for you letter of March 25. Let me deal with your points in turn.

1. We shall be in touch with Larry Sulak towards the end of the exhibition. When we know which images are available and have chosen 4 of them, we shall mail a cheque to D Saupe.

2. We shall resolve this with Larry Sulak.

3. I shall follow this up.

4. We look forward to the post cards. They are not here yet.

5. The video will be completed next week and i shall have a 3/4 in. copy sent to H O Peitgen. We shall not allow it to be used anywhere apart from in our exhibition.

6. Enclosed is an invitation to the opening. An invitation to Mrs Nancy Hollomon is going out in the post today. Doug Smith has been sent an invitation already.

7. We shall not use the slides you made for Robert for publicity although we have been asked for photos by many newspapers and magazines who have received our press release (copy enclosed). We shall send them some of Robert's pictures instead.

I have been writing the detailed picture labels and noticed that the 'guiding picture' of the locations of the Mandelbrot set blow-ups you mentioned in your text seems to be absent. Did you forget to include it or did you not get around to making it? Is it connected with the poster you put together showing how the Julia sets vary around the Mandelbrot set? I would be grateful if yu could send it to me as soon as possible if it exists or if you can create it.

I too enjoyed working with you and think this will be a well attended and important exhibition. I hope you settle back into German life smoothly.

Yours sincerely

Oliver

Oliver Strimpel

Peter Richter

Fachbereich Physik
Universitaet Bremen
D-2800 Bremen 33
Fed. Rep. of Germany

March 25, 1986

To: Oliver Strimpel
Computer Museum Boston
Museum Wharf
300 Congress Street, Boston, MA
Tel. (617) 423-6758

Re: Colors of Chaos

Dear Oliver,

I thought it might be a good idea to write up the things on which we agreed in connection with our exhibition. Please let me know if there are items on which you disagree or which you might want to add. I will refer to the Bremen group as MAPART. Consider me as there representative back home, and keep contact with Drs. Dietmar Saupe or Heinz-Otto Peitgen over here; their address is Dept. of Mathematics, University of California at Santa Cruz, Santa Cruz, CA 95405.

1. MAPART has provided 12 pictures for the Colors of Chaos exhibit, six of which are owned by the Physics Department of Boston University. Please make sure that they will be handed over to the Physics Department c/o Larry Sulak, Chairman, when the exhibit is over. Four of the remaining pictures will be purchased by the Computer Museum, at a price of $ 75.- a piece. Please send your check of $ 300.- to D. Saupe. The last two pictures may either be sold for the same amount or returned to Mrs. Nancy Hollomon, 121 Carlton St., Brookline, MA 02146, Tel. 734-4763.

2. The Computer Museum has paid the framing of 14 pictures. The cost for 8 will be reimbursed by the Physics Department of BU (they have already two of the eight pictures).

3. The Computer Museum has obtained 960 copies of the Mandelbrot Set poster that I produced with BU Graphics Arts. So far these posters have been paid by the Physics Department (but I do not know the amount). Please check with them for appropriate reimbursement.

4. The Museum Shop ordered 2000 postcards at a price of $ 500.- MAPART. They should be arriving from Bremen shortly; a bill will be included.

5. The Museum copied and edited a part of the computer graphical film that was produced by H.-O. Peitgen and D. Saupe together with Radio Bremen. You will not use the original music. A copy will be sent to MAPART for approval. Their Copyright extends to your version of the film. It will not be used other than for the exhibit.

6. The Museum keeps us informed about important events in connection with the exhibition. Even though we cannot attend the opening, e.g., we would like to receive copies of the invitation cards. Also please keep in mind that under no circumstances do we want to jeopardize the upcoming exhibit at the Museum of Science. Therefore, please keep in touch with Doug Smith of their exhibits department, and invite him to major events. I would also appreciate if you could invite Mrs. Nancy Hollomon (my adopted mother).

7. I agreed that you obtain copies of the slides that I gave Robert Devaney. These are only for your personal use. Please understand that due to a lot of trouble we had in the past, I must insist on this clause.

It was a pleasure working with you, and I hope the show will be a success. Whenever people are interested in getting in touch with us, please refer them to MAPART - either in Bremen or in Santa Cruz. Our book should come out some time in June.

Good luck! Sincerely yours,

Peter

Peter H. Richter

To Oliver

Date 3/31 Time 11:45

WHILE YOU WERE OUT

M Sally

of B.Y.U

Phone 353 - 9365

Area Code Number Extension

TELEPHONED	X	PLEASE CALL	X
CALLED TO SEE YOU		WILL CALL AGAIN	
WANTS TO SEE YOU		URGENT	
RETURNED YOUR CALL			

Message regarding the bill for framing the Mandelbrot sets

Wants to speak w/ you

Bill

Operator

Boston University

College of Liberal Arts
111 Cummington Street
Boston, Massachusetts 02215

Department of Mathematics

February 24, 1986

Mr. Oliver Strimpel
Computer Museum
300 Congress Street
Boston, Massachusetts

Dear Oliver:

Here are some possible little blurbs which could be posted near my pictures. Do you want more? I liked your version of the "Colors of Chaos" very much, so I think I'll ask you to edit these too!

By the way, I have finally gotten approval to produce the second film, this time in Hollywood. I hope to have it by the beginning of April if all goes well. We start computing March 3 on a Cray at Digital Productions.

Regards,

Bob

Robert L. Devaney

RLD:lv

The
Computer
Museum

300 Congress Street
Boston, MA 02210
(617) 426-2800

April 25 1986

Professor Robert Devaney
Mathematics Department
Boston University
111 Cummington Street
Boston
MA 02215

Dear Robert

First of all let me thank you again for giving such an excellent and highly appropriate lecture at the opening of the exhibit. Several people, some of whom have been quite critical of previous lectures, said how much they enjoyed it.

I forgot to return the tape to you at the lecture - we had already finished with it by then. I enclose it now. Do keep me informed about the new material!

I also enclose a handful of free admissions to the museum for you to dispose of as you wish.

Many thanks again for all your involvement. Do let us know if you stop by over the course of the summer.

Yours sincerely

Olive

Oliver Strimpel

enclosures: video tape, admission passes

12 March 1986

Oliver Strimpel
Curator
The Computer Museum
300 Congress Street
Boston, MA 02210

Dear Mr. Strimpel:

I am enclosing an updated copy of our DEMO program with updated data files. Please destroy all copies of the previous program and data files to avoid confusion. During the show, we would like you to give credit for the images as follows:

These images were prepared using:

FractalMagic

Available from: Sintar Software
P. O. Box 3746
Bellevue, WA 98009
206-455-4130

We will try to prepare a DEMO program that displays more images for your show, but I'm going to be in China for two weeks starting this Friday, so I don't know if we will have enough time.

We hope that your patrons like the show.

Yours truly,

Mark

Mark W. Bolme, President
SINTAR SOFTWARE

February 18 1986

Mark W Bolme, President
Sintar Software
PO Box 3746
Bellevue
WA 98006-3746

Dear Mr Bolme

Thank you for your letter of 10 February 1986.

We are delighted that you may be able to make us a demo version of your mandelbrot set program and look forward to seeing it. We intend to run it on an IBM PC with an expansion unit, effectively making the combination into an IBM PC XT. There is a color graphics adapter, 384K of memory, 8087 chip, but no EGA card.

Please get in touch if you need any further information.

Yours sincerely

Oliver Strimpel

10 February 1986

Oliver Strimpel
Curator
The Computer Museum
300 Congress Street
Boston, MA 02210

Dear Mr. Strimpel:

Regretably, our program is not fast enough to use in an interactive exhibit. Computational realities interfere.

However, I believe that we CAN come up with a demo version that will make an eye-catching exhibit. I will be shipping you a preliminary copy for your inspection in about 10 days.

In the meantime, can you tell me a few things about your hardware? You will NEED an IBM PC compatible with a Color Graphics Adapter, a color monitor, 256K and one disk drive. The following are not necessary, but may be useful:

- extra memory,
- EGA card,
- hard disk,
- 8087 chip.

384K

Yours truly,

Mark W. Bolme

Mark W. Bolme, President
SINTAR SOFTWARE

5 February 1986

Mark Bolme
Token Software
POB 3746
Belleview
WA 98009

Dear Mr Bolme

I read about your program 'Mandelzoom' in Jerry Pournelle's column in the January 1986 issue of Byte Magazine. Having written a couple of programs to generate Mandelbrot Sets, I know that it is a computationally intensive process, and I wonder how quickly you are able to bring up freshly computed zooms.

We are mounting and exhibit we are calling 'Colors of Chaos' in which we shall be showing large color prints of various sets, including the Mandelbrot Set as well as sets generated using trigonometric functions. These were generated in part by the Bremen group whose images were featured in Scientific American and also in the exhibition 'Frontiers of Chaos'. We had hoped to have an interactive version running, allowing visitors to roam around the complex plane themselves, discovering new structures. However we couldn't see a way to do this quickly enough to provide a satisfactory response for a museum visitor. Do you think your program might be suitable as an exhibit? 'Colors of Chaos' will be open to the public from April 10 till June 15 1986.

We would be interested in seeing a copy of your program and wonder if you might send us a copy? If we use it in the exhibit, we will certainly credit the source.

I enclose the latest issue of the Computer Museum Report to give you some background on the Museum. We are a public non-profit collections-based Museum, and have 30,000 square feet of exhibits with about 30 hands-on interactive exhibits. Please let us know if you would like to know more about the Museum. I look forward to hearing from you and thank you for any help you might offer.

Yours sincerely

Oliver Strimpel
Curator

Telephone Conversation with 'Sally' from Physics Dept, BU.

Struck a deal: Museum gets 1000 M-Set posters (cost $255) in exchange for frames for 8 physics dept images ($190 cost to Museum).

No cash need change hands.

OJ 4/8/86

COLORS OF CHAOS VIDEO

Number	Title	Comments
0	~~Experience Fractal Images and Forntiers of Chaos~~ Colors of Chaos Exploring ~~Julia fractals geometry with Julia and~~ the Mandelbrot set	
5	~~10~~ Experiments	
13	~~Green blobs start~~	cut wholes sequence

Start

1 72	1-10 Voyage into Seahorse Valley Millionfold Zoom into the Mandelbrot set	good; need to say this is blow-up on the Mandelbrot Set
114	~~2-10 Orbiting over Jupiter 4~~	Images not in exhibit; extraneous sphere - cut (Julia)
159	3-10 Rolling Kites	Color sweep through real pl.

Vickie Marsh

hyperbolic-type curves. Too long and repetitive; cut to 50%

2 ✓ 186 ~~4-10 Following Potentials~~ — Beautiful; say what is going on?

Stem Detail of the tail of a ~~$~~ sea horse.

Tail of a Seahorse

+ detail

227 5-10 Rams and Whirls — Color sweeps into and then out of spirals; too long: do some forward and some backward; say what is going on? Julia

3 ✓ 260 6-10 ~~Unseen~~ Deep Sea — Color sweep about central spiral. Where is it? M-Set? What's going on? Too long - need cutting

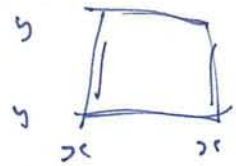
The bottom of a nigh

Neighboring valley to ~~seahorse valley~~.

286	7-10 ~~Dragonflies~~ in Metamorph.	A bit dill - cut completely or at least by 80%	real plane
~~323~~	8-10 Ganymed 3 Orbiting over Jupiter 4	Cut	Jubila
~~332~~	9-10 Fly Lor	Cut	
4 ✓ 354	10-10 Secco Lightenings Flashes of Lightening ~~The Cintacles of the~~ .... Color Schism	M-Set with tendrils showing and color sweep. Good but a bit long - could use some explanation	
✓ 374	Endpiece	CRT, fingers on keyboard: quite nice	
✓ 383	Oblongs + Credits	Credits all ok?	

ELIZABETH PETROSKEY
276-4445

edited for the exhibit
COLORS OF CHAOS
at The Computer Museum, Boston



123 45 Exp
1 2 3 4 Julia
Sin z
1 2 3 4 5 6

De Vaney.

Dynamical System Text

1. 2 exponential function.
2. 3 $\pi i e^z$
   Period doubling
   Exponential
   } exp family.
   } Use as intro.

3. 4 Julia Sets. sin z Siegel Disk phenomenon.

   Coefficients.

function
window
number of iterations

6 of them
4. Sin z ; series in the film
   $(1+ci)\sin z$

Transition from black to colored region.

function
parameter value
.. ..

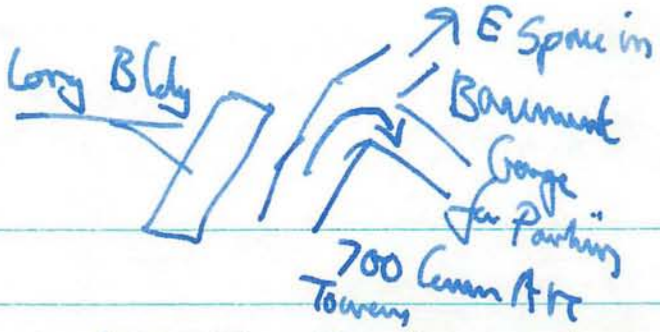
Good explanation of color scheme.

Frontiers of Chaos

Peter Richter Prints

Catalog

Has Maps 2, 7, 23, 25, 33, 47, 56 multi-globe version, 65, 66



And 4 more not in catalog   Typically 20 by 16.

Total no of pictures 13.

Possibly joint with Robert Devaney, Chairman of Math Dept at BU.

353 2560 Math Dept

Film - showing changing parameters 16mm

Shuman exhibition in March.
c. 4wks

Boston Science Museum: Doug Smith. 723 2501 x 287
Planner, Exhibits Dept

Miriam Rosy -
Pati 703 951 2805
Rosy coming up. 201 757 6749
Inexpensive hotel 201 754 9178 6-9pm
Can drive out

near WBUR
Dorms
Behind Warren Towers
111 Cummington St
Rm 140 B

Richter

5 : Mand Set.

1 Mand Set of Magnetic function } Same sign

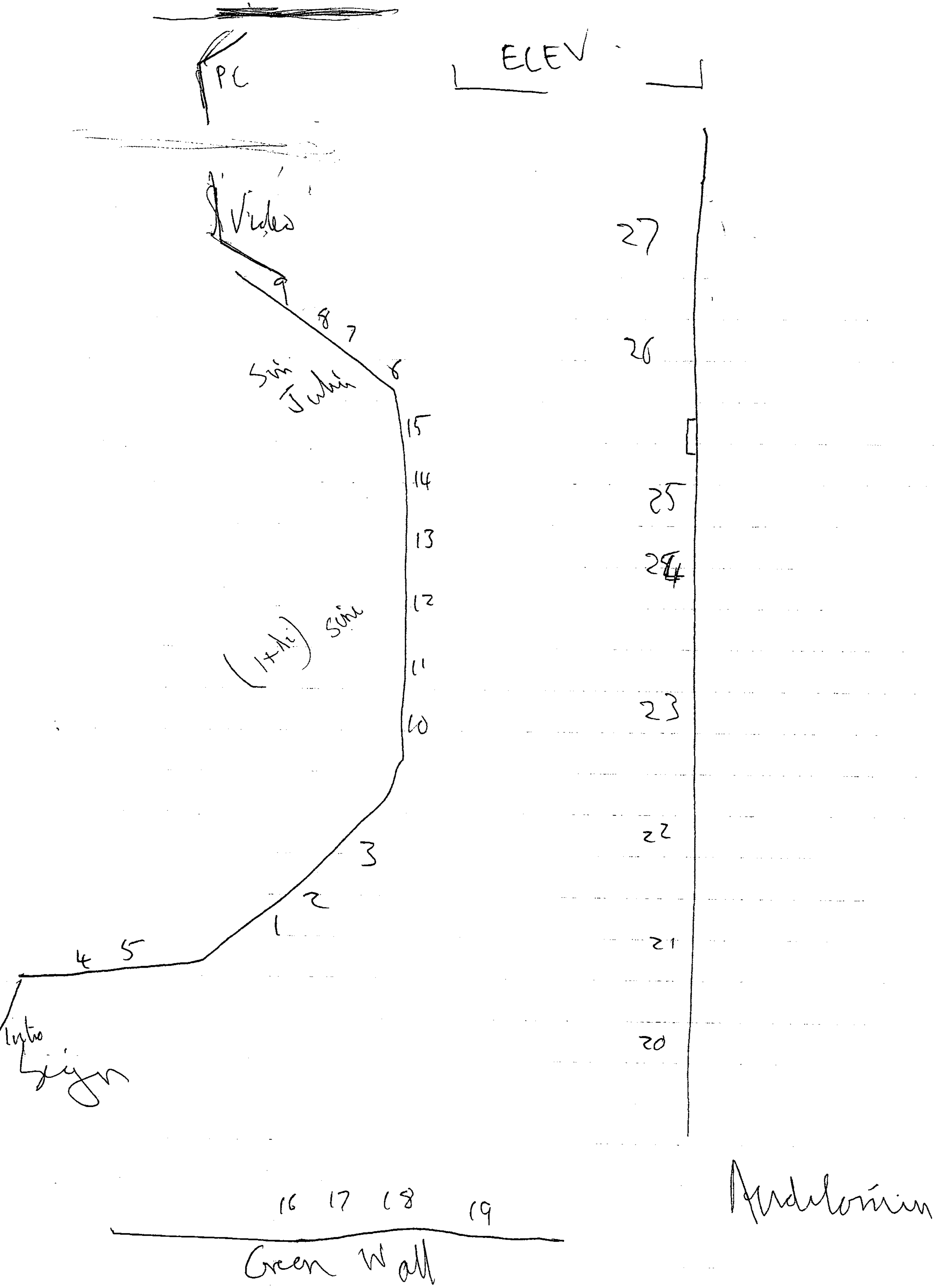
1 Julia from Orig Mand. (quadratic)

1 " " Magnetic Model.

3 ~~#~~: Studies of Newton's Root-finding. } Sign.

Test :

1 Deriving Cole Volterra Equation. Sign.


PC
ELEV
Video
9
8
7
6
15
14
13
12
11
10
3
2
1
4
5
Intro
Sign
27
26
25
23
22
21
20
16
17
18
19
Green Wall

COLORS OF CHAOS

The pictures shown here are the results of using computer graphics as a tool for research in complex dynamics, a branch of mathematics and physics. The goal is to understand what happens to simple mathematical formulae when they are iterated. Iteration means evaluating a formula again and again, each time using the previous value of the formula as the new value of the variable. In physics, this process describes the way natural systems evolve.

These computer experiments, combined with mathematical intuition, uncover universal patterns and stimulate the progress of mathematics. They are also important in the new field of fractal geometry (described in The Computer and the Image gallery) and in the understanding of nonlinear processes in nature.

The pictures are built up in the following way: each point in the picture is iterated using the mathematical formula under investigation. For example, if the formula is the trigonometrical function cosine, this corresponds to entering the number corresponding to a point in the picture into a calculator, and pressing the cosine button again and again. The point in the picture is then colored depending on what happens. In some of the pictures, the color shows how quickly the point "escapes" to infinity, leaving black those points that never escape. In others, the colors show where points end up under iteration, with the shades indicating how quickly they get there.

It takes a lot of computing to produce a high resolution image because each point is iterated up to several thousand times. Indeed, Robert Devaney is now using a Cray supercomputer. A low resolution image can be made in a few hours on a personal computer.

The images form two series:

Julia Sets and Mandelbrot Sets generated by the iteration of polynomial functions and ratios therof by a team from the University of Bremen led by Heinz-Otto Peitgen and Peter Richter.

Iterations of sine, cosine and the exponential function by Robert L Devaney from the Department of Mathematics at Boston University.

Further Reading

The Beauty of Fractals by H-O Peitgen and P H Richter, Springer-Verlag 1986

The Fractal Geometry of Nature by Benoit B Mandelbrot, W H Freeman 1983

Introduction to Chaotic Dynamical Systems by Robert L Devaney, Benjamin Cummings 1985

Scientific American Computer Recreations column by A K Dewdney August 1985 issue

Telephone Conversation with 'Sally' from Physics Dept, BU.

Struck a deal: Museum gets ~~x~~ 1000 M-Set posters (cost $255) in exchange for frames for 8 physics dept images ($190 cost to Museum).

No cash need change hands.

OJ 4/8/86

Larry Sulak, Chairman
Physics Dept. Boston Univ.
Tel. 353 – 9365
9454

re Prints belonging to phys is dept.

Peter Richter
Fachbereich Physik, Universität Bremen
28 BREMEN 33
Tel. 01149 - 421 - 218 - 3680

Dietmar Saupe (representing MAPART)
Dept. of Mathematics, Univ. of California at S.C.
Santa Cruz, CA 95064

The Frame Gallery, Inc.

Framing: 300 Harvard Street [rear]
Brookline, MA. 02146
617 - 232 - 2070

NO 5517 DATE 2/7/86 DUE DATE 2/21/86

NAME Computer Museum CLERK JM FRAMER

ADDRESS Oliver Stinbeck

PHONE (daytime) 426-2800

ARTWORK INCLUDED ☑ YES ☐ NO NUMBER OF PIECES ______ FORMAT ☐ V ☐ H

DESCRIPTION ~~M. 29 7/8 x 25 3/4~~

SIZE OF FRAMED WORK ______ V ______ H N. 29 1/2 x 25 5/8

a

FRAME ☐ WOOD ☐ METAL ☐ CLIPS

TYPE

☐ CHOP ☐ FIT ☐ ASSEMBLE ONLY

M. Gold 11 28.90

N. Silver 11 28.90

b

☑ GLASS ☑ REGULAR

☐ PLEXIGLAS ☐ NON GLARE

☐ NO GLASS M. 15.60

☐ SHRINK WRAP N. 15.60

☐ LAMINATE

c

MOUNT ☐ DRY ☐ COLD ☐ STRETCH

ON

d

MAT ☐ SINGLE ☐ DOUBLE OUTSIDE SIZE ______ V ______ H

1 (inner)

T 3

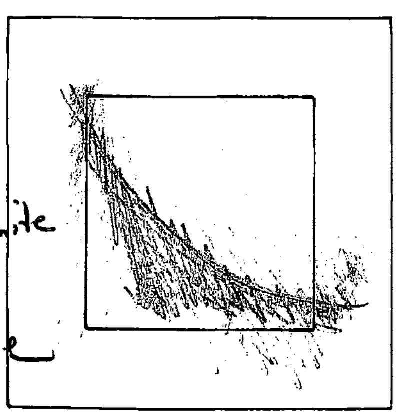
☐ MAT ______ V S 3

☐ FLOAT ______ H B 3

14.65 M. Alpha photo White

1. Cut 1/8 off top

Bottom + Right side



2 (outer)

T ______

______ V S ______

______ H B ______

e

OTHER N. Alpha photo White
14.65

We are not responsible for work held for more than 60 days.

a	57.	80	subtotal	118.30
b	31.	20	tax	
c	29.	30	total	
d			deposit	
e			balance	59.15

Arrows on back point to direction for framing make sure this is transferred to back of mount for fitting direction

also some photos are marked B.U. physic dept. this too must be transfered to the back of finished work

19 1/4 x 23 1/4
6 6
$\boxed{25\ 1/4 \times 29\ 1/4}$ x 2

The Frame Gallery, Inc.

waiting on mat board

Framing: 300 Harvard Street [rear]
Brookline, MA. 02146
617 - 232 - 2070

NO 5517 DATE 2/7/86 DUE DATE 2/21/86

NAME Computer Museum CLERK JM FRAMER

ADDRESS Oliver

PHONE (daytime)

ARTWORK INCLUDED ☐ YES ☐ NO NUMBER OF PIECES ____ FORMAT ☐ V ☐ H

DESCRIPTION

SIZE OF FRAMED WORK ____ V ____ H G. 20 x 20 1/4 J. 20 x 20 H. 19 3/4 x 19 5/8

a

FRAME ☐ WOOD ☐ METAL ☐ CLIPS

TYPE

☐ CHOP ☐ FIT ☐ ASSEMBLE ONLY

G. ~~[illegible]~~ Gold 11 21.50 x 3
H. Silver 11
J. Gold 11

b

☑ GLASS ☑ REGULAR
☐ PLEXIGLAS ☐ NON GLARE
☐ NO GLASS G. 3 x 9.90
☐ SHRINK WRAP H
☐ LAMINATE J

c

MOUNT ☐ DRY ☐ COLD ☐ STRETCH

ON

d

MAT ☐ SINGLE ☐ DOUBLE OUTSIDE SIZE ____ V ____ H

1 (inner) 2 (outer)

☐ MAT ____ V T 2 ____ V T ____
☑ FLOAT ____ H S 2 ____ H S ____
B 2 B ____

G Photo White Alpha
H
J 3 x 9.50

e

OTHER Arrows on back point direction for framing to make sure this is transferred to back of mount for fitting direction

50%
no Tax

a	64.	50	subtotal	122.70
b	29.	70	tax	
c	28	50	total	
d			deposit	
e			balance	61.35

We are not responsible for work held for more than 60 days.

15 3/4 x 15 3/4
4 4

J. 19 3/4 x 19 3/4

also some photos are marked B.U physics Dept. this too must be transferred to the Back of finished work.

15 3/4 x 16
4 4

G. 19 3/4 x 20

15 1/2 x 15 1/2
4 4

H. 19 1/2 x 19 1/2

Heinz - Otto Peitgen
Dept. of Mathematics
Univ. of California at Santa Cruz
Santa Cruz CA 95064
Tel. (408) 429 - 2718

please send him copy of film

To Oliver

Date 2.24 Time 1:50.

WHILE YOU WERE OUT

M Jeff

of Frame Gallery

Phone

Area Code Number Extension

TELEPHONED	X	PLEASE CALL	X
CALLED TO SEE YOU		WILL CALL AGAIN	
WANTS TO SEE YOU		URGENT	

RETURNED YOUR CALL	

Message

all photos are completed.

Balance is: $322.51

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February 18 1986

Robert Devaney
Chairman, Mathematics Department
111 Cummington Street
Boston University
Boston MA 02215

Dear Robert

Please find enclosed my effort at the introductory text. I would welcome comments from both you and Peter (to whom I am also sending a copy) on how it might be improved.

I have put all the credits at the end. If you think it more appropriate to list the 'artists' at the top, that is fine with me.

I look forward to hearing from you. Any further news on the video front?

Yours sincerely

Oliver Strimpel

Oliver Strimpel

The
Computer
Museum

300 Congress Street
Boston, MA 02210

(617) 426-2800

February 18 1986

Peter Richter
Physics Department
Boston University
Boston
MA 02215

Dear Peter

I enclose my effort at the introductory text for the Colors of Chaos exhibit. I would be pleased to hear any comments you might have on how to improve it. I have also sent a copy to Robert Devaney.

I have viewed the video tape. It needs a little cutting down, perhaps by 50%. I will have to book a session at a studio out at DEC to do a good job of this. Can you let me keep the tape for another week or two? We will also then have the opportunity to change the text on the video. Perhaps we could discuss this. If you have the time, perhaps you could come out with me to the studio for the edit session.

Let's be in touch when you get this. I shall be in all week except Wednesday when (in the afternoon) you can reach me at home at 526-7423.

Yours sincerely

Oliver

Oliver Strimpel

Frontiers fo Chaos
Mathmatical experiments done at the Center for Complex Dynamics
Univ. of Bremen, Fed. Rdp. of Germanyu

The field of Complex Dynmanics is currently one of the most active areas of research are the tools of mathematical experimentation - a new discipline that makes new suggestions and brings up questions to stimulate the progress of mathematic. The common theme of all pictures in this exhibition is the competition between centers of attraction. A simple feedback process carries a point towards one or another of a number of centers. Colors indicate how for from its centre near the boundaries of the competing domains. Some of the most startling pictures analyze the fate of these boundfaries as parameters are varied.

The aim of our computer studies is to detect general features in this complex plane which at first sight seems beyond imagination. Combined with mathematical insight however, experiments of this kind are apt to uncover "universal scenarios" in the transition between different types of behavior, and therby to deepen our understanding of natural nonlinear processes.

Introduction to Chaotic Dynamical Systems

Robert L. Devaney

Benjamin - Cummings
2727 Sand Hill Rd.
Menlo Park Calif 94025

Frontiers of Chaos (to be renamed)

Proposal for a Temporary Exhibit

Introduction

Frontiers of Chaos is a graphic exhibit of some 50 images that has been replicated by the Goethe Institute and is touring in Europe and America, booked till well into 1987. I saw it first in the Exploratorium in July during SIGGRAPH. The Boston Science Museum has booked it for January 1987.

The images were computed by a group at the University of Bremen, one of whom (Peter Richter) is visiting BU this year. They show computer graphic renderings of Mandelbrot Sets, Julia Sets and some other mathematical functions. They are all based on a similar type of idea, that of iteration of a complex function. The images can appeal on several levels:

1. Pretty pics
2. How can I do these on my pc?
3. What's the meaning/maths behind this?

Content of CM Exhibit

Peter Richter happens to have 14 prints (unframed), many of which are in the FOC exhibit, all of which are interesting.

The Chairman of the maths dept at BU, Robert Devaney has also produced similar images (iterating exp, cos, sin as opposed to the squaring function) which, though not as well colored or as sharp as the Bremen images, are still good enough to show. He also has a 2 minute 16mm film clip showing what happens as you change parameters - beautiful patterns that mimic the onset of turbulence.

I propose we put together a joint exhibition consisting of

1. 14 Bremen images from Peter Richter
2. approx 14 of Robert Devaney's pictures
3. Video tape of Robert Devaney's film
4. Something interactive, possibly on the Amiga showing generation of Julia sets using graphics and sound. This could then replace/supplement the IBM pc fractal demo in the image gallery.

The exhibit would be introduced with several text panels. Each image would need a small label, along the lines of the Byte covers.

Timescale and Budget

Open April 15 approx and close if/when SIGGRAPH '85 art show comes, but not sooner than July.

Costs:

Framing	500	
Text panels	600	
Video tape	200	(use existing player, control box and monitor)
Interactive	500	
Miscellaneous	200	

Total	2000

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ITERATION OF SIN(Z)

The complex function $\sin z$ has a rather interesting structure when as a dynamical system, i.e., when iterated. These pictures detail the dynamical behavior of $\sin z$ as well as $(1 + \lambda i) \sin z$ for various values of the parameter λ. The colored regions in these pictures indicate points which "tend" to ∞ under iteration of these maps. The black regions contain those points which remain bounded under iteration. One may show that the colored regions are actually the Julia sets of the maps and so contain all of the chaotic dynamics, while the black regions are the stable sets. Note how the black region for $\sin z$ (the first picture) explodes into color as the parameter is varied.

The colors in these pictures simply tell how quickly a point tend to ∞ under iteration. Red points tend to ∞ most quickly, followed by points colored in shades of orange, and then followed by yellow, green, blue, and violet.

THE EXPLODING SINE FUNCTION

This series of pictures shows how the Julia set of a complex function may "explode". Depicted are Julia sets of a constant times the sine function for various values of the constant. The first picture is that of the ordinary sine. In successive steps, we make the constant complex but still near 1. Note how sine has a large black basin of attraction with a fractal boundary. As the constant changes, this basin is destroyed - colors invade the basin and gradually squeeze the black regions into smaller and smaller spiral regions. These changes - called bifurcations by mathematicians - are the subject of much contemporary research.

The above paragraph deals with the series of six pictures that will be displayed, as well as with the film clip.

THE JULIA SET

These pictures describe the barest outline of a geometric object known as the Julia set, named after the French mathematician Gaston Julia who studied these objects in the 1920's. While Julia succeeded in discovering many of the properties of the Julia set, he never lived to see the fascinating geometric structures of these sets as disclosed by computer graphics.

Iteration

These pictures are generated by a process called iteration. A grid of points in the plane (usually 400 by 400) is selected. To each point in this grid, we then apply a particular mathematical procedure over and over again, using the results of the previous computation as the seed for the next. This iteration is performed up to 100 times on each point in the grid. Thus each picture is generated by a total of 16×10^6 iterations ($400 \times 400 \times 100$).

THE ALGORITHM

These pictures are produced by asking the question, which points tend to infinity under iteration. We color a point depending upon how many iterations it takes for the point to move far away. Red points move to infinity most quickly, followed by orange, yellow, green, blue, indigo, and violet. Black points never go to infinity. The colored regions represent the mathematical object known as the Julia set.

FACILITIES

These pictures were generated on an AED 512 color graphics screen with a program generated by Chris Small, Chris Mayberry, and Sherry Smith, all undergraduates at Boston University. The computations were done on an IBM 3081 mainframe. The typical picture uses between 4 and 8 minutes of CPU time.

FRACTALS

Many of these pictures are fractals. A fractal is a geometric object with a complicated shape which tends to look the same under successive magnifications.

Note: these first three are more technical than the rest.

THE JULIA SET

These pictures describe the barest outline of a mathematical object called a Julia set, after the French mathematician, Gaston Julia, who first studied these sets in the early twentieth Century. For functions like $\sin z$ (as depicted here), or $\cos z$, or $\exp z$, the Julia set is the closure of the set of points which tend to ∞ under iteration of the function. Equivalently, the Julia set may also be characterized as the set of expanding "periodic points." As you can see, the topological structure of these sets is quite intricate. For the mathematician, understanding the structure of these sets is a basic goal: since these functions are among the most elementary transcendental functions, a detailed knowledge of their Julia sets is a necessary first step in the understanding of the more complicated equations that occur in nature.

DYNAMICAL SYSTEMS

Dynamical Systems is a branch of Mathematics which studies the evolution in time of discrete or continuous processes. Examples of discrete processes include the growth and decline of the population of a biological species or the daily fluctuations of the Stock Market. Examples of continuous processes include the motion of the planets or the changes in the weather. The Mathematician is interested in understanding the long term behavior of these processes, i.e., will the population of the species tend to die out or will it explode in the far distant future?

How meaningful?

To study such processes, the Mathematician often constructs simplified mathematical models of the physical systems, and then studies these models instead. In these pictures, we have tried to detail the dynamics of several extremely simple dynamical systems: iteration of the complex analytic functions $\sin z$, $\cos z$, and $\exp z$.

Siegel disks

The mathematical object depicted in Figure F is a Siegel disk. This is the large black region in the center of Figure F. The Siegel disk is the subject of intensive mathematical investigation. Note how the black region explodes into color after just a small change in the parameter: the Siegel disk is a highly unstable object.

Figure F $\quad (.6 + .8i) \sin z$

$$\text{XMIN} = -3$$
$$\text{YMIN} = -3$$
$$\text{SIDE} = 6$$
$$\text{ITER} = 100$$

Figure G $\quad (.6 + .81i) \sin z$

$$\text{ITER} = 100$$

Figure H $\quad (.6 + .81i) \sin z$

$$\text{XMIN} = -2$$
$$\text{YMIN} = -1$$
$$\text{SIDE} = 2$$
$$\text{ITER} = 255$$

Figure I $\quad (.6 + .81i) \sin z$

$$\text{XMIN} = -.5$$
$$\text{YMIN} = -.5$$
$$\text{SIDE} = 1$$
$$\text{ITER} = 255$$

Fractal Patterns

These images represent the Julia sets of various members of the exponential family, i.e. $(c + di) \exp z$ where c and d are parameters. For different values of c and d, this family contains an infinite variety of different patterns.

Figure C	$\pi i \exp z$
Figure D	$-(3 + i) \exp z$
Figure E	$(-5.2 + 8.33i) \exp z$

The Complex Exponential

Each of these pictures represents the Julia set of .375 $\exp(z)$. (One is twisted 90^o to give the face-like effect). The prominent black swirls represent points which have not escaped before a certain maximum number of iterations (75 in Figure A, 250 in Figure B). Changing the number of iterations allows the experimenter to begin to comprehend the dynamics of the system.

Figure A .375 $\exp(z)$

$$\begin{aligned} \text{XMIN} &= 0 \\ \text{YMIN} &= -.5 \\ \text{SIDE} &= 2 \\ \text{ITER} &= 75 \end{aligned}$$

Figure B .375 $\exp(z)$

$$\begin{aligned} \text{XMIN} &= -.5 \\ \text{YMIN} &= -1.3 \\ \text{SIDE} &= 2.6 \\ \text{ITER} &= 250 \end{aligned}$$

The Exploding Sine Function

Figure J depicts the Julia set of the complex sine function. All of the "infinite snowmen" – the black regions in this picture – consist of points which tend to 0 under iteration. On the other hand, the colored regions represent points which tend to infinity. Note how this region "explodes" as the parameter is varied. The basin of attraction of 0 is destroyed by a small change in the parameter.

Figure J: $\sin z$

$$\text{XMIN} = -4$$
$$\text{YMIN} = -4$$
$$\text{SIDE} = 8$$
$$\text{ITER} = 35$$

Figure K $(.1 + .17i)\sin z$

$$\text{XMIN} = -3$$
$$\text{YMIN} = -3$$
$$\text{SIDE} = 6$$
$$\text{ITER} = 35$$

Figure L: $(1 + .24i)\sin z$

Figure M: $(1 + .35i)\sin z$

Figure N: $(1 + .6i)\sin z$

Boston University

College of Liberal Arts
111 Cummington Street
Boston, Massachusetts 02215

Department of Mathematics

Bob 353-2560

April 23, 1986

Mark Hunt
Computer Museum
300 Congress Street
Boston, Massachusetts 02210

Dear Mark:

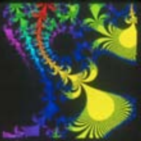
Enclosed are four slides that might be helpful to you for PR purposes. Would you please return these slides when you are finished with them?

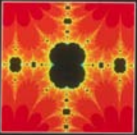
Sincerely,

Robert L. Devaney

Robert L. Devaney

RLD:lv





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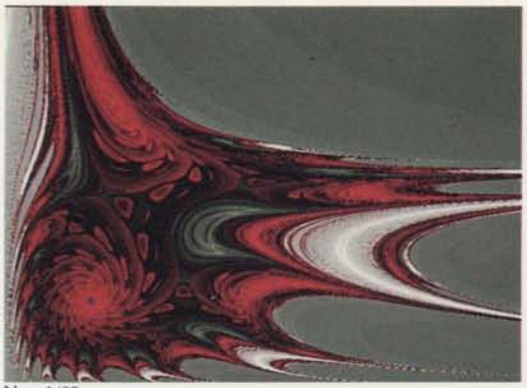

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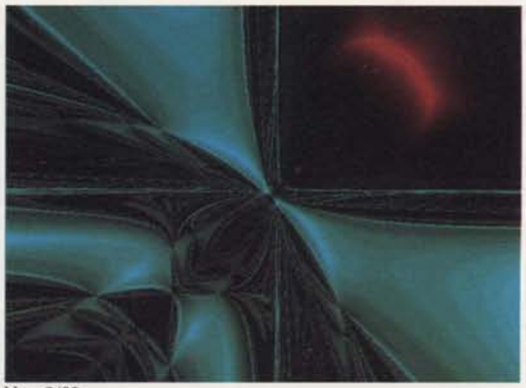

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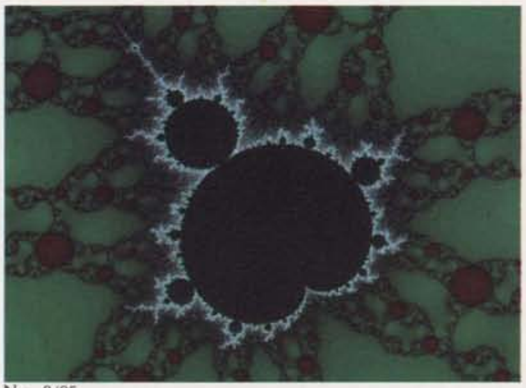

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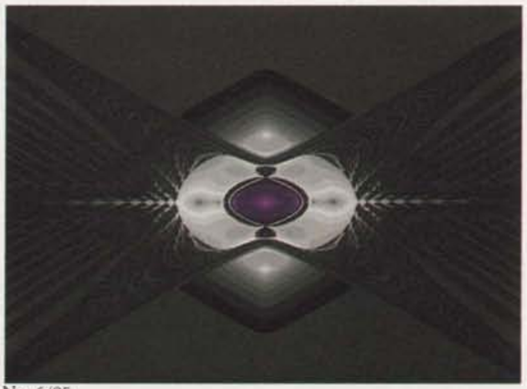

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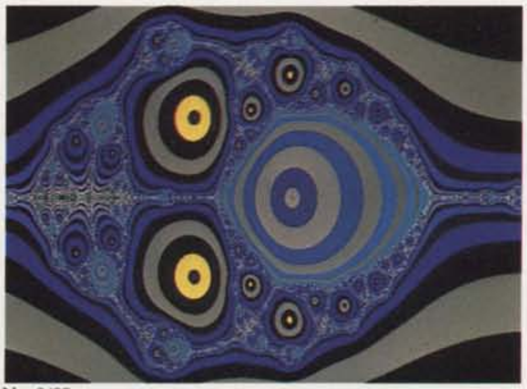

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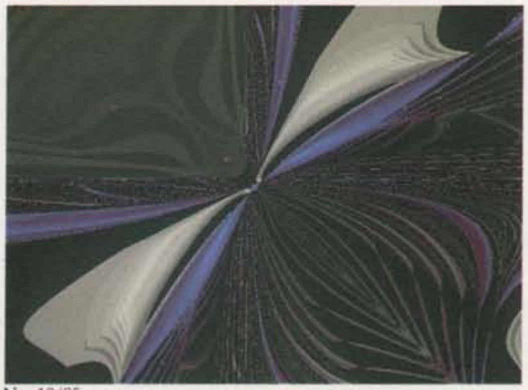

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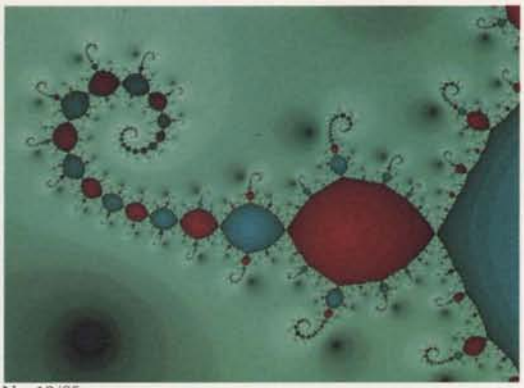

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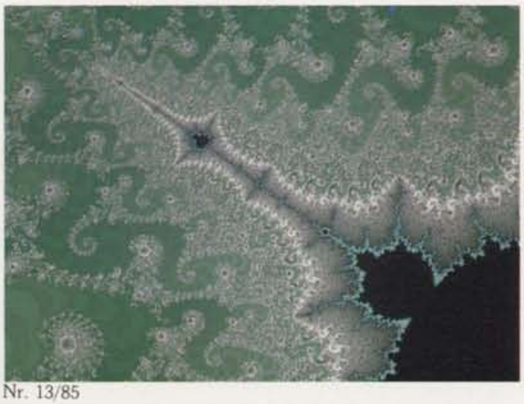

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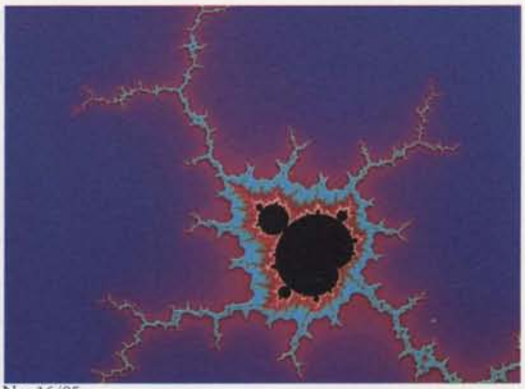

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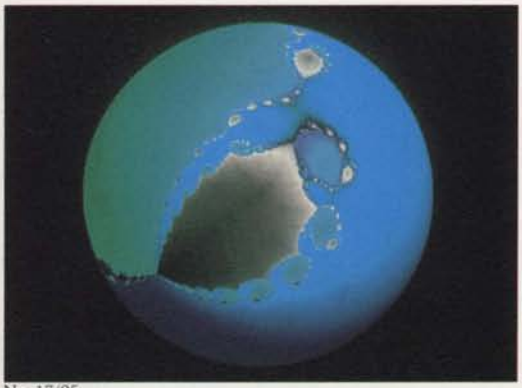

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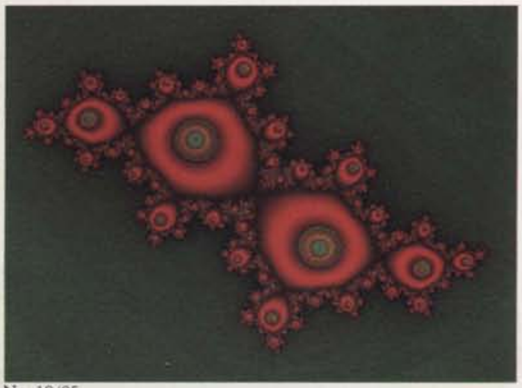

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FRONTIERS OF CHAOS · COLOR COMPUTER GRAPHICS

Als Mathematiker und Physiker studieren wir Prozesse, in denen sich nach einfachen Gesetzen komplexe Strukturen entfalten. Uns interessiert die Frage, ob in dem Wechselspiel zwischen Chaos und Ordnung universelle Szenarien verborgen sind: Prototypen von Musterbildung in Raum und Zeit, die in der scheinbar grenzenlosen Vielfalt des Lebens und Denkens Orientierungspunkte sein könnten.

Auf der Suche nach der Antwort gilt es immense Datenmengen zu bewältigen. Diese Arbeit übernimmt der Computer. Es gilt aber auch, das Verborgene sichtbar zu machen. Dazu bedarf es mathematischer Einsicht und physikalischer Intuition, die durch kaum ein einziges Hilfsmittel jemals so befördert wurde wie derzeit durch die interaktive Computer-Grafik.

Der unmittelbare sinnliche Zugang, den dieses neue Medium eröffnet, erschließt den Objekten unserer Forschung zugleich ein neues Publikum. Die Faszination, die wir angesichts der natürlichen Schönheit unserer Bilder empfinden, scheint auf geheimnisvolle Weise auch solche Betrachter zu ergreifen, denen die Inhalte eher fremd sind.

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Our team of mathematicians and physicists is engaged in research on the development of complex structures on the basis of simple underlying laws. We are looking for universal scenarios in the interplay between order and chaos, for prototypes of pattern formation in space and time. The bewildering complexity of nature - and science - calls for paradigms.

To achieve these goals one has to process enormous amounts of data. This task keeps our computer busy day and night. Then comes the need to visualize the hidden structures. Here we rely on mathematical insight and physical intuition - but in order to put them into effect there has never been invented a more powerful tool than interactive computer graphics.

This new technique has had a fundamental impact on experimental mathematics. It has added a new dimension to our perception of natural and artificial phenomena. Its creative potential has transformed scientific statements into images whose fascination attracts both the expert and the layman.

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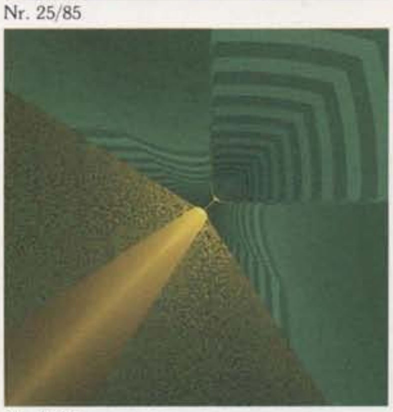

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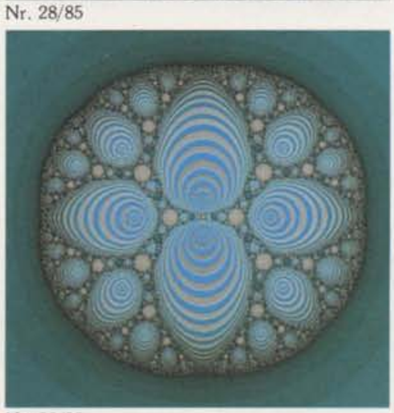

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EXPLORING THE MANDELBROT SET

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August 1985

COMPUTER RECREATIONS

A computer microscope zooms in for a look at the most complex object in mathematics

by A. K. Dewdney

The Mandelbrot set broods in silent complexity at the center of a vast two-dimensional sheet of numbers called the complex plane. When a certain operation is applied repeatedly to the numbers, the ones outside the set flee to infinity. The numbers inside remain to drift or dance about. Close to the boundary minutely choreographed wanderings mark the onset of the instability. Here is an infinite regress of detail that astonishes us with its variety, its complexity and its strange beauty.

The set is named for Benoit B. Mandelbrot, a research fellow at the IBM Thomas J. Watson Research Center in Yorktown Heights, N.Y. From his work with geometric forms Mandelbrot has developed the field he calls fractal geometry, the mathematical study of forms having a fractional dimension. In particular the boundary of the Mandelbrot set is a fractal, but it is also much more.

With the aid of a relatively simple program a computer can be converted into a kind of microscope for viewing the boundary of the Mandelbrot set. In principle one can zoom in for a closer look at any part of the set at any magnification [*see cover of this issue and illustrations on pages 17–19*]. From a distant vantage the set resembles a squat, wart-covered figure eight lying on its side. The inside of the figure is ominously black. Surrounding it is a halo colored electric white, which gives way to deep blues and blacks in the outer reaches of the plane.

Approaching the Mandelbrot set, one finds that each wart is a tiny figure shaped much like the parent set. Zooming in for a close look at one of the tiny figures, however, opens up an entirely different pattern: a riot of organic-looking tendrils and curlicues sweeps out in whorls and rows. Magnifying a curlicue reveals yet another scene: it is made up of pairs of whorls joined by bridges of filigree. A magnified bridge turns out to have two curlicues sprouting from its center. In the center of this center, so to speak, is a four-way bridge with four more curlicues, and in the center of these curlicues another version of the Mandelbrot set is found.

The magnified version is not quite the same Mandelbrot set. As the zoom continues, such objects seem to reappear, but a closer look always turns up differences. Things go on this way forever, infinitely various and frighteningly lovely.

Here I shall describe two computer programs, both of which explore the effects of iterated operations such as the one that leads to the Mandelbrot set. The first program generated the colored illustrations appearing in this month's column. The program can be adapted to run on personal computers that have the appropriate hardware and software for generating graphics. It will create satisfying images even if one has access only to a monochrome display. The second program is for readers who, like me, need an occasional retreat from infinite complexity to the apparent simplicity of the finite.

The word "complex" as used here has two meanings. The usual meaning is obviously appropriate for describing the Mandelbrot set, but the word has a second and more technical sense. A number is complex when it is made up of two parts, which for historical reasons are called real and imaginary. These terms no longer have any special significance: the two parts of a complex number might as well be called Humpty and Dumpty. Thus $7 + 4i$ is a complex number with real part 7 (Humpty) and imaginary part $4i$ (Dumpty). The italic i next to the 4 shows which part of the complex number is imaginary.

Every complex number can be represented by a point in the plane; the plane of complex numbers is called the complex plane. To find $7 + 4i$ in the complex plane, start at the complex number 0, or $0 + 0i$, and measure seven units east and four units north. The resulting point represents $7 + 4i$. The complex plane is an uncountable infinity of such numbers. Their real parts and their imaginary parts can be either positive or negative and either whole numbers or decimal expansions.

Adding or multiplying two complex numbers is easy. To add $3 - 2i$ and $7 + 4i$, add the parts separately; the sum is $10 + 2i$. Multiplying complex numbers is only slightly more difficult. For example, if the symbol i is treated like the x in high school algebra, the product of $3 - 2i$ and $7 + 4i$ is $21 + 12i - 14i - 8i^2$. At this stage a special property of the symbol i must be brought into play: it happens that i^2 equals -1. Thus the product can be simplified by collecting the real and the imaginary parts: it is $29 - 2i$.

It is now possible to describe the iterative process that generates the Mandelbrot set. Begin with the algebraic expression $z^2 + c$, where z is a complex number that is allowed to vary and c is a certain fixed complex number. Set z initially to be equal to the complex number 0. The square of z is then 0 and the result of adding c to z^2 is just c. Now substitute this result for z in the expression $z^2 + c$. The new sum is $c^2 + c$. Again substitute for z. The next sum is $(c^2 + c)^2 + c$. Continue the process, always making the output of the last step the input for the next one.

Strange things happen when the iterations are carried out for particular values of c. For example, here is what happens when c is $1 + i$:

first iteration, $1 + 3i$
second iteration, $-7 + 7i$
third iteration, $1 - 97i$

Note that the real and the imaginary parts may grow, shrink or change sign. If this process of iteration continues, the resulting complex numbers get progressively larger.

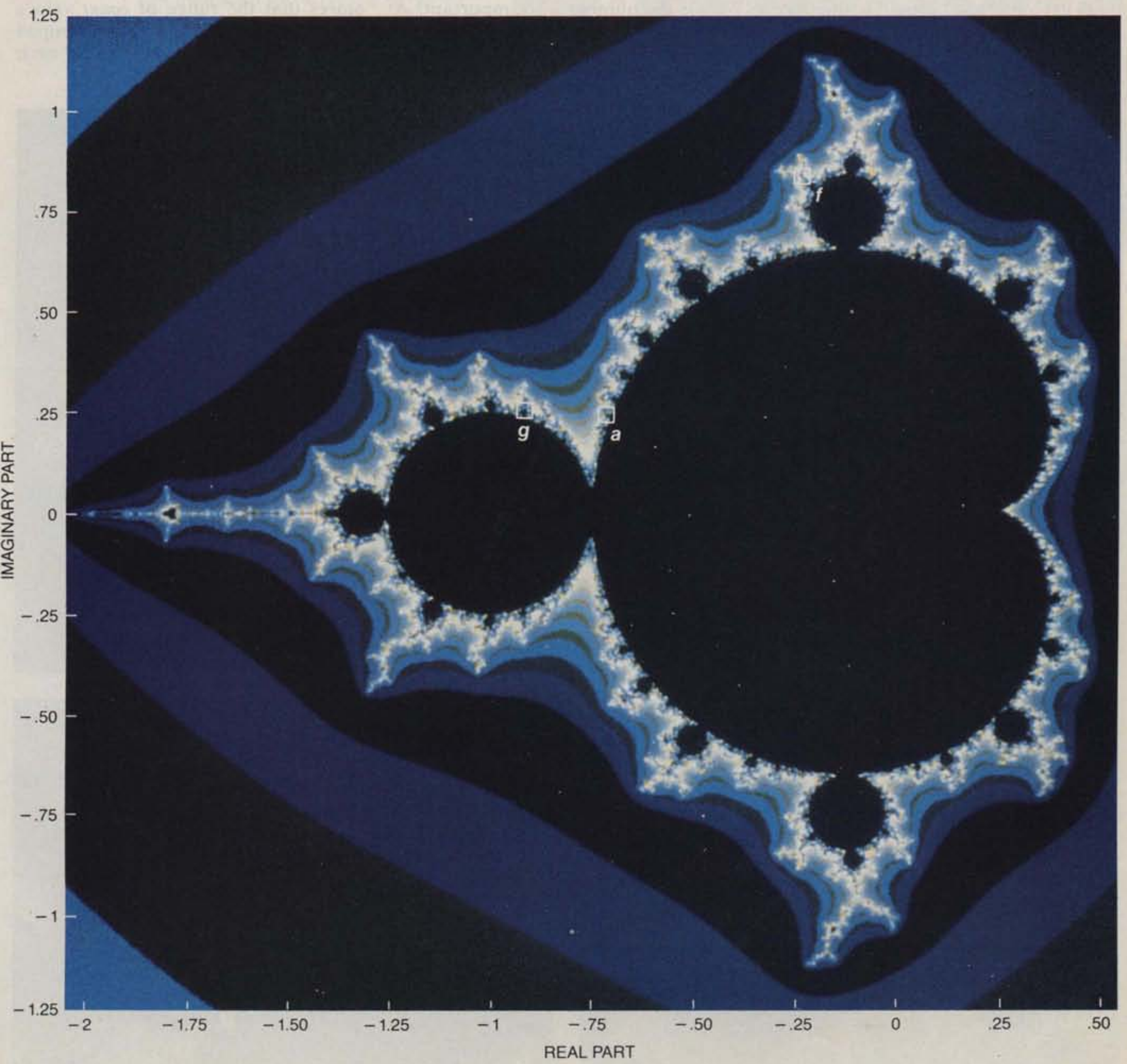
What exactly is meant by the size of a complex number? Since complex numbers correspond to points in the plane, ideas of distance apply. The size of a complex number is just its distance from the complex number 0. That distance is the hypotenuse of a right triangle whose sides are the real and the imaginary parts of the complex number. Hence to find the size of the number square each of its parts, add the two squared values and take the square root of the sum. For example, the size of the complex number

$7 + 4i$ is the square root of $7^2 + 4^2$, or approximately 8.062. When complex numbers reach a certain size under the iterative process I have just described, they grow very quickly: indeed, after a few more iterations they exceed the capacity of any computer.

Fortunately I can ignore all the complex numbers c that run screaming off to infinity. The Mandelbrot set is the set of all complex numbers c for which the size of $z^2 + c$ is finite even after an indefinitely large number of iterations. The program I am about to describe searches for such numbers. I am indebted in all of this to John H. Hubbard, a mathematician at Cornell University. Hubbard is an authority on the Mandelbrot set, and he was one of the first people to make computer-generated images of it. Most of the images in this article were made by Heinz-Otto Peitgen and his colleagues at the University of Bremen. Peitgen learned the art from Hubbard.

Hubbard's program has inspired a program I call MANDELZOOM. The program sets up an array called *pic*, which is needed for saving pictures. The entries of *pic* are separate picture elements called pixels, which are arranged in a grid pattern. Hubbard's array has 400 columns and 400 rows, and Peitgen's is even larger. Readers who want to adapt MANDELZOOM for personal use must choose an array suited to their equipment and temperament. Larger arrays impose a longer wait for the pictures, but they improve the resolution.

In the first part of MANDELZOOM one may select any square region of the complex plane to be examined. Specify the southwest corner of the square with the complex number to which it corresponds. Two variables in the program, *acorner* and *bcorner*, enable one to enter the real part and the imaginary part of the number respectively. Specify the length of each side of the



The Mandelbrot set and its coordinates in the complex plane. The details shown on the cover and on the next two pages are outlined square by entering a value for a variable called *side*.

The second part of the program adjusts the array *pic* to match the square of interest by computing the size of a variable called *gap*. *Gap* is the distance within the square between adjacent pixels. To obtain *gap* divide *side* by the number of rows (or columns) in *pic*.

The heart of the program is its third part. Here a search is made for the complex numbers *c* in the Mandelbrot set, and colors are assigned to the numbers that are, in a special sense, nearby. The procedure must be carried out once for every pixel; thus Hubbard's 400-by-400 array requires 160,000 separate computations. Assume the program is currently working on the pixel in row *m* and column *n*; the third part then breaks down into four steps:

1. Calculate one complex number *c* that is assumed to represent the pixel: add $n \times gap$ to *acorner* to obtain the real part *ac* of *c*; add $m \times gap$ to *bcorner* to obtain the imaginary part *bc* of *c*. It is not necessary to include the imaginary number *i* in the program.

2. Set a complex variable *z* (which has parts *az* and *bz*) equal to $0 + 0i$. Set an integer variable called *count* equal to 0.

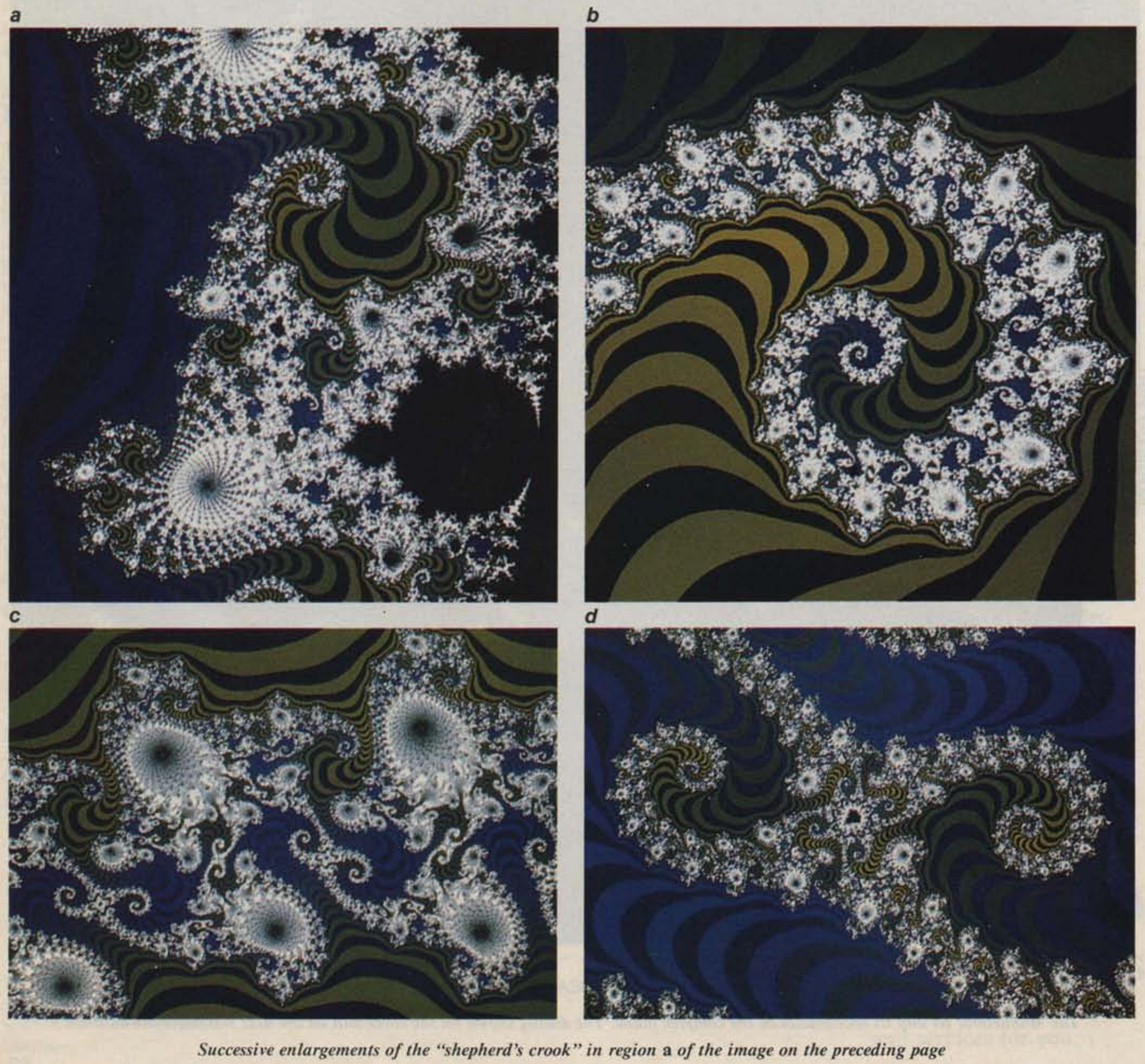
3. Carry out the following three steps repeatedly, until either the size of *z* exceeds 2 or the size of *count* exceeds 1,000, whichever comes first:

$$z \leftarrow z^2 + c$$
$$count \leftarrow count + 1$$
$$size \leftarrow \text{size of } z$$

Why is the number 2 so important? A straightforward result in the theory of complex-number iterations guarantees that the iterations will drive *z* to infinity if and only if at some stage *z* reaches a size of 2 or greater. It turns out that relatively many points with an infinite destiny reach 2 after only a few iterations. Their slower cousins become increasingly rare at higher values of the variable *count*.

4. Assign a color to *pic* (*m*,*n*) according to the value reached by *count* at the end of step 3. Display the color of the corresponding pixel on the screen. Note that the color of a pixel depends on only one complex number within its tiny domain, namely the one at its northeast corner; the behavior of this number then represents the behavior of the entire pixel.

The scheme for assigning colors requires that the range of *count* values attained within the array be grouped into subranges, one subrange for each



Successive enlargements of the "shepherd's crook" in region a *of the image on the preceding page* color. Pixels for which the size of z reaches 2 after only a few iterations are colored red. Pixels for which the size of z reaches 2 after relatively many iterations are colored violet, at the other end of the spectrum. Pixels for which the size of z is less than 2 even after 1,000 iterations are assumed to lie in the Mandelbrot set; they are colored black.

It makes sense to leave the colors unspecified until the range of *count* values in a particular square has been determined. If the range is narrow, the entire color spectrum can then be assigned within that range. Thus Hubbard suggests that in step 4 only the value of *count* be assigned to each array element of *pic*. A separate program can then scan the array, determine the high and low values of *count* and assign the spectrum accordingly. Readers who get this far will certainly find workable schemes.

The reader who does not have a color monitor can still take part in black and white. Complex numbers for which z is larger than 2 after r iterations are colored white. The rest are colored black. Adjust r to taste. To avoid all-night runs the array can be, say, 100 rows by 100 columns. Hubbard also suggests it is perfectly reasonable to reduce the maximum number of iterations per point from 1,000 to 100. The output of such a program is a suggestive, pointillistic image of its colored counterpart [*see illustration on next page*].

How powerful is the "zoom lens" of a personal computer? It depends to some degree on the effective size of the numbers the machine can manipulate. For example, according to Magi (my microcomputer amanuensis at the University of Western Ontario), the IBM PC uses the 8088 microprocessor, a chip manufactured by the Intel Corporation designed to manipulate 16-bit numbers. A facility called double precision makes it possible to increase the length of each number to 32 bits. With such double precision Magi and I calculate that magnifications on the order of 30,000 times can be realized. Higher precision software that in effect strings these numbers together can enhance the numerical precision to hundreds of significant digits. The magnification of the Mandelbrot set theoretically attainable with such precision is far greater than the magnification needed to resolve the nucleus of the atom.

Where should one explore the complex plane? Near the Mandelbrot set, of course, but where precisely? Hubbard says that "there are zillions of beautiful spots." Like a tourist in a

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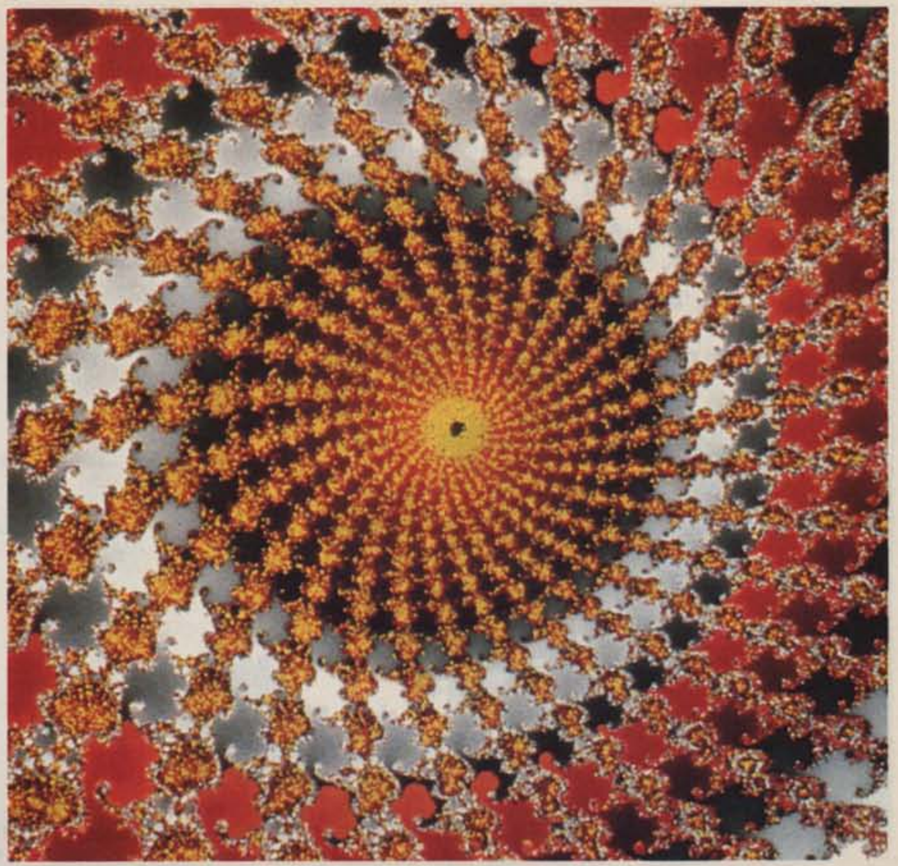
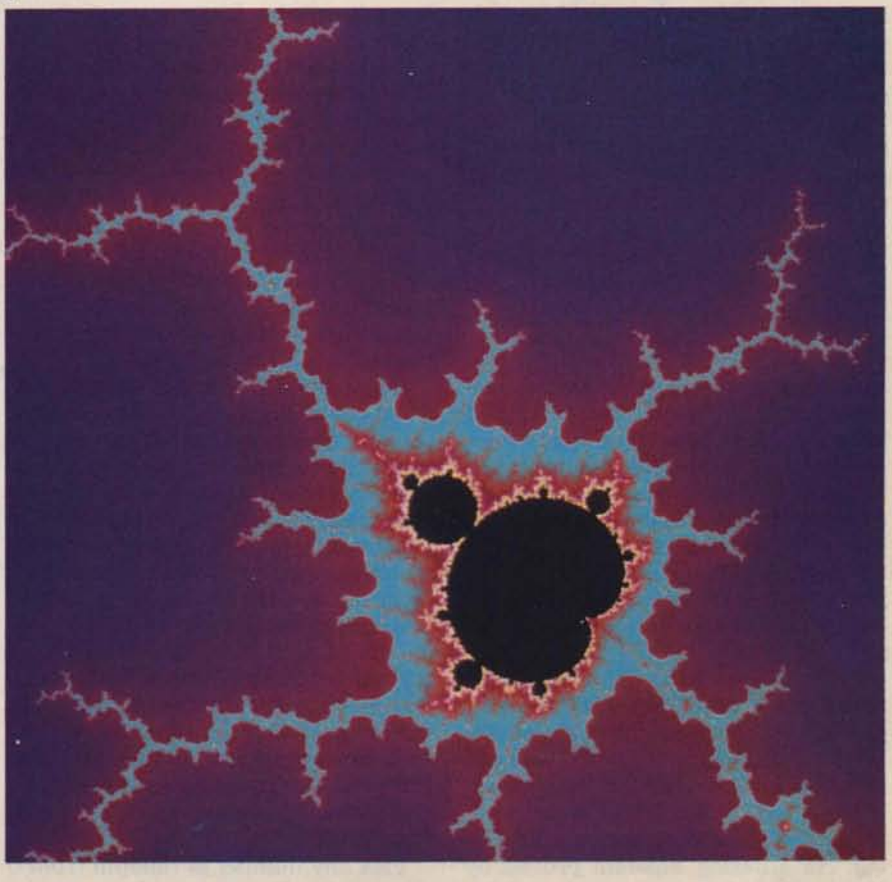


A compound eye peering out from a region of image d *on the opposite page*

f



A miniature Mandelbrot in region f *on page 17, tethered to the main set by a filament*

Pointillist, miniature Mandelbrot generated by a monochrome monitor

land of infinite beauty, he bubbles with suggestions about places readers may want to explore. They do not have names like Hawaii or Hong Kong: "Try the area with the real part between .26 and .27 and the imaginary part between 0 and .01." He has also suggested two other places:

Real Part	*Imaginary Part*
−.76 to −.74	.01 to .03
−1.26 to −1.24	.01 to .03

The reader who examines the color images accompanying this article should bear in mind that any point having a color other than black does not belong to the Mandelbrot set. Much of the beauty resides in the halo of colors assigned to the fleeing points. Indeed, if one were to view the set in isolation, its image might not be so pleasing: the set is covered all over with filaments and with miniature versions of itself.

In fact none of the miniature Mandelbrots are exact copies of the parent set and none of them are exactly alike. Near the parent set there are even more miniature Mandelbrots, apparently suspended freely in the complex plane. The appearance is deceiving. An amazing theorem proved by Hubbard and a colleague, Adrian Douady of the University of Paris, states that the Mandelbrot set is connected. Hence even the miniature Mandelbrots that seem to be suspended in the plane are attached by filaments to the parent set. The minatures are found almost everywhere near the parent set and they come in all sizes. Every square in the region includes an infinite number of them, of which at most only a few are visible at any given magnification. According to Hubbard, the Mandelbrot set is "the most complicated object in mathematics."

Readers with a simple appetite for more color images of the Mandelbrot set and other mathematical objects can write to Hubbard for a brochure (Department of Mathematics, Cornell University, Ithaca, N.Y. 14853). The brochure includes an order form with which one can buy 16-inch-square color prints that are similar in quality to the Peitgen images shown here.

Confronted with infinite complexity it is comforting to take refuge in the finite. Iterating a squaring process on a finite set of ordinary integers also gives rise to interesting structures. The structures are not geometric but combinatorial.

Pick any number at random from 0 through 99. Square it and extract the last two digits of the result, which must also be a number from 0 through 99. For example, 59^2 is equal to 3,481; the last two digits are 81. Repeat the process and sooner or later you will generate a number you have already encountered. For example, 81 leads to the sequence 61, 21, 41 and 81, and this sequence of four numbers is then repeated indefinitely. It turns out that such loops always arise from iterative processes on finite sets. Indeed, it is easy to see there must be at least one repeated number after 100 operations in a set of 100 numbers; the first repeated number then leads to a loop. There is a beautiful program for detecting the loops that requires almost no memory, but more of this later.

It takes only an hour to diagram the results of the squaring process. Represent each number from 0 through 99 by a separate point on a sheet of paper. If the squaring process leads from one number to a new number, join the corresponding points with an arrow. For example, an arrow should run from point 59 to point 81. The first few connections in the diagram may lead to tangled loops, and so it is a good idea to redraw them from time to time in such a way that no two arrows cross. A nonintersecting iteration diagram is always possible.

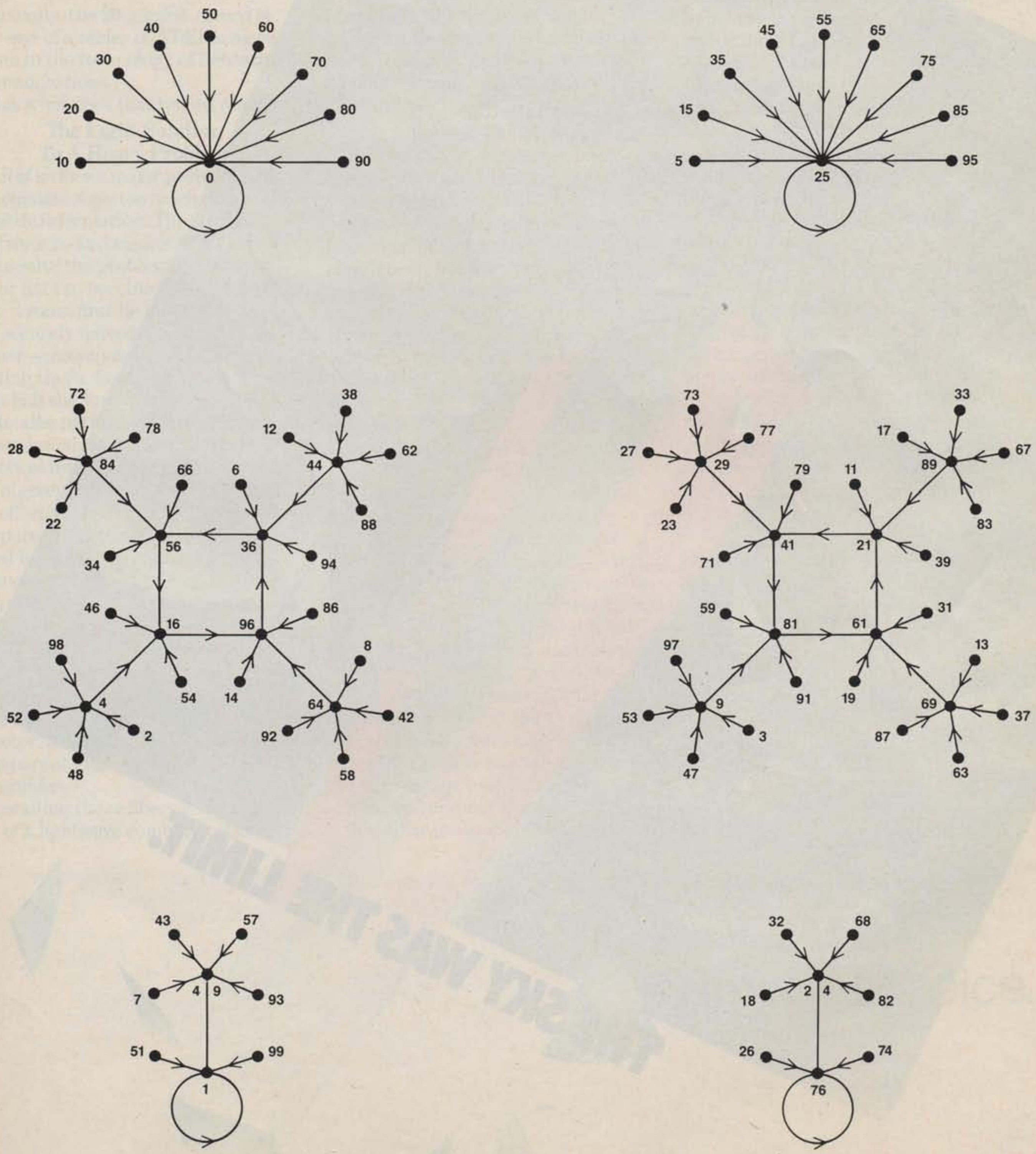
One can go even further. Separate subdiagrams often arise, and they can be displayed in a way that highlights some of the symmetries arising from the iterations. For example, the nonintersecting iteration diagram for the squaring process on the integers from 0 through 99 includes six unconnected subdiagrams. The pieces come in identical pairs and each piece is highly symmetrical [*see illustration on opposite page*]. Can the reader explain the symmetry? What would happen if the integers from 0 through 119 were used instead? Is there a relation between the number of unconnected pieces found in the diagram and the largest integer in the sequence?

Similar patterns of iteration hold for some of the complex numbers in the Mandelbrot set: for certain values of c repeated iterations of $z^2 + c$ can lead to a finite cycle of complex numbers. For example, the complex number $0 + 1i$ leads to an indefinite oscillation between the two complex numbers $-1 + 1i$ and $0 - 1i$. The cycle may even have only one member. Whether such cycles are found in a finite set or in the infinite Mandelbrot set, they are called attractors.

Each of the six parts of the iteration diagram for the integers 0 through 99 includes one attractor. Geometrically the attractor can be represented as a polygon, and the sets of numbers that lead into it can be represented as trees.

One way to find an attractor by computer is to store each newly generated number in a specially designated array. Compare the new number with all the numbers previously stored in the array. If a match is found, print all the numbers in the array from the matching number to the number just created. The method is straightforward and easy to program. Nevertheless, it can take a long time if the array is large. An attractor cycle within an array that includes n numbers would take on the order of n^2 comparisons to discover: each new number must be compared with up to n numbers in the array.

There is a clever little program that will find an attractor much faster. The program requires not n words of memory but only two, and it can be encoded on the simplest of programmable pocket calculators. The program is found in a remarkable book titled *Mathematical Recreations for the Programmable Calculator,* by Dean Hoffman of Auburn University and Lee Mohler of the University of Alabama. Needless to say, many of the topics that are covered in the book can be



The six components of the iteration diagram for squaring the first 100 integers readily adapted to computer programs.

The program is called RHOP because the sequence of numbers that eventually repeats itself resembles a piece of rope with a loop at one end. It also resembles the Greek letter rho (ρ). There are two variables in the program called *slow* and *fast*. Initially both variables are assigned the value of the starting number. The iterative cycle of the program includes just three instructions:

$$fast \leftarrow fast \times fast \pmod{100}$$
$$fast \leftarrow fast \times fast \pmod{100}$$
$$slow \leftarrow slow \times slow \pmod{100}$$

The operation mod 100 extracts the last two digits of the products. Note that the squaring is done twice on the number *fast* but only once on the number *slow*. *Fast* makes its way from the tail to the head of the rho twice as fast as *slow* does. Within the head *fast* catches up with *slow* by the time *slow* has gone partway around. The program exits from its iterative cycle when *fast* is equal to *slow*.

The attractor is identified by reiterating the squaring process for the number currently assigned to *slow*. When that number recurs, halt the program and print the intervening sequence of numbers.

I should be delighted to see readers' diagrams that explore the effects of iterative squaring on finite realms of varying size. The diagrams can be done on a computer or by hand. Discrete iteration is a newly developing mathematical field with applications in computer science, biomathematics, physics and sociology. Theorists might watch for a book on the subject by François Robert of the University of Grenoble.

The two-dimensional beings who inhabit the planet Arde are deeply grateful to the many readers who tried to improve the crossover circuit I described in May. That circuit is made up of 12 two-input *nand*-gates. I asked readers to find the minimum number of *nand*-gates—and *nand*-gates only—from which a crossover circuit can be built. Most of the circuits submitted have 10 gates, a mild improvement, but three readers found an eight-gate crossover [*see illustration on this page*].

In the eight-gate circuit there is one three-input *nand*-gate and two single-input *nand*-gates. The latter act as inverters, converting a 0 signal into a 1 signal and vice versa. The three readers who discovered the eight-gate solution are Eric D. Carlson of Cambridge, Mass., Dale C. Koepp of San Jose, Calif., and Steve Sullivan of Beaverton, Ore. I have passed their names along with the improved crossover circuit to my Ardean friends. Believe it or not, the same crossover circuit appears under U.S. Patent 3,248,573 (April 26, 1966). Robert L. Frank, who is a systems consultant in Birmingham, Mich., wrote that the patent was awarded to Lester M. Spandorfer of Cheltenham, Pa., Albert B. Tonik of Dresher, Pa., and Shimon Even of Cambridge, Mass.

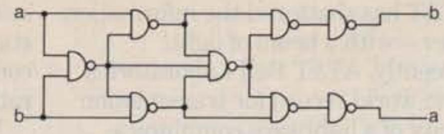
It seems natural to wonder whether the circuit actually appears in any present-day device. It is also natural to wonder whether there is an even smaller *nand* crossover. One supposes not.

C. Walter Johnson of Long Beach, Calif., wrote to me describing a wide variety of planar circuits that incorporate several types of gate. Apparently it is possible to build not only crossover circuits in two dimensions but also planar flip-flops. The flip-flops provide memory for a two-dimensional computer.

One-dimensional computers in the form of cellular automata have been investigated by Stephen Wolfram of the Institute for Advanced Study in Princeton. It is too early to say what contributions readers may have made to this field after reading Wolfram's "Glider Gun Guidelines," but I can pass along some initial reactions. A sin of commission sent a few readers off chasing gliders in the line automaton code-numbered 792. Wolfram and I meant to specify code 357. A sin of omission was my decision not to mention the line automata known to be capable of universal computation. I thought of describing such a line automaton, first constructed by Alvy Ray Smith in 1970. At the time Smith was a graduate student at Stanford University. I was afraid that the description of Smith's universal line automaton would unduly complicate the article: the automaton has 18 states ($k = 18$) and three-cell neighborhoods ($r = 1$).

Arthur L. Rubin of Los Angeles has made a sensible suggestion for defining the speed of light in an arbitrary line automaton. Rubin's suggestion corrects a defect in an earlier definition that sets the speed of light equal to one cell per unit of time. The old definition ignores the possibility that not all automata can attain such speeds. The revised speed of light is "the maximum speed of propagation of any impulse (say to the right)." The leading edge of the impulse is defined by the condition that only 0's can lie to its right. Rubin goes on to prove that the speed of light is 1/3 for the line automaton code-numbered 792.

In my May column I also asked whether the line automaton called



A crossover circuit with eight **nand**-*gates*

Ripple has a one-way glider gun. Gliders fired from such a gun would spew out unendingly to the right but never to the left. William B. Lipp of Milford, Conn., has made a simple and charming argument against the existence of such a gun. "Consider a pattern," he writes, "that never has nonzero values to the left of some block labeled 0. Observe that the leftmost nonzero value in the pattern must always be a 1. If it were a 2, the 2 would ripple to the left forever, thus contradicting the assumption that no nonzero entries lie to the left of block 0. But the leftmost 1 must become 0 on the next cycle, moving the left boundary of the pattern at least one block to the right." Thus either a glider ripples to the left or its gun is eaten away by 0's.

Other readers sought to show that Ripple is not capable of universal computation. For some automata one can prove a sufficient condition, namely that the halting problem is decidable. Ripple halts when all its cells contain 0's, but the halting conditions for any universal computing machine it might contain could be quite different.

Several readers attempted constructions of line automata capable of universal computation, among them Frank Adams of East Hartford, Conn., Jonathan Amsterdam of Cambridge, Mass., Kiyoshi Igusa of Brandeis University and Carl Kadie of East Peoria, Ill. The constructions are all straightforward and believable, but Kadie, not content with his one-dimensional automaton, went on to suggest a zero-dimensional one. It would consist of a single cell, and it seems reasonable to call it a point automaton. Readers with a theoretical bent might enjoy pondering the universality of a point automaton. Is it possible?

Alvy Ray Smith published his proof for the existence of a computation-universal line automaton in 1971, in *Journal of the Association for Computing Machinery*. One might have thought a career with such an auspicious start would today be blossoming in some well-known academic institution. Instead it has blossomed in a quite different setting: Smith is director of computer-graphics research for Lucasfilm, Ltd., in San Rafael, Calif. In a future column I hope to report on some of the amazing cinematic effects produced at the Lucasfilm laboratory.